# Doctoral Qualifying Exam:
# Linear Algebra and Numerical Analysis.

Wednesday, September 2, 1998.

You have three hours for this exam. Show all working in the books provided.

**1.** (a) Write the differential equation $y'' + y = 0$ as a first order system in the form $du/dt = A\ u$. Use the eigenvalues and eigenvectors of the matrix $A$ to compute the solution of the differential equation $y'' + y = 0$ with the initial conditions $y_0 = 2$ and $y_0' = 0$.

(b) Find the Jordan form of the matrix $A$ given by

$$A = \begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 3 \end{pmatrix}.$$

**2.** (a) If $V$ is the subspace spanned by the vectors $\mathbf{u}_1$, $\mathbf{u}_2$, and $\mathbf{u}_3$, where

$$\mathbf{u}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{u}_3 = \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix},$$

find a matrix $A$ that has $V$ as its row space and a matrix $B$ that has $V$ as its nullspace.

(b) If $A$ is a square matrix, show that the nullspace of $A^2$ contains the nullspace of $A$. Show also that the column space of $A^2$ is contained in the column space of $A$.

**3.** (a) Let $A$ be an invertible $n \times n$ matrix. If $A = L_1 D_1 U_1$ and $A = L_2 D_2 U_2$, prove that $L_1 = L_2$, $U_1 = U_2$, and $D_1 = D_2$. Start by deriving the equation $L_1^{-1} L_2 D_2 = D_1 U_1 U_2^{-1}$.

(b) Prove that the expression for any vector in a vector space in terms of the vectors in a basis of the space is unique.

**4.** Derive Simpson's rule for approximating $\int_a^b f(x)dx$ by using a quadratic Lagrange polynomial.

(a) What is the order of the error?

(b) Using Taylor series, derive Simpson's rule with a higher order term.

**5.** (a) Apply Newton's method to the function

$$f(x) = \begin{cases} \sqrt{x}, & x \geq 0 \\ -\sqrt{-x}, & x \leq 0 \end{cases}$$

which has the unique root $\alpha = 0$. What is the behavior of the iterates? Do they converge, and if so, at what rate?

(b) Do the same as in (a) but with

$$f(x) = \begin{cases} \sqrt[3]{x^2}, & x \geq 0 \\ -\sqrt[3]{x^2}, & x \leq 0 \end{cases}$$

(c) Explain your answers to (a) and (b).

**6.** (a) Assume that that the matrix $\mathbf{A} \in R^{n \times n}$ is singular, so that the system $\mathbf{A}\mathbf{x} = \mathbf{b}$ does not have a unique solution. Consider the matrix splitting scheme $\mathbf{A} = \mathbf{B} + (\mathbf{A} - \mathbf{B})$, where $\mathbf{B}$ is nonsingular. Show that the corresponding iterative scheme $\mathbf{x}_{n+1} = (\mathbf{I} - \mathbf{B}^{-1}\mathbf{A})\mathbf{x}_n + \mathbf{B}^{-1}\mathbf{b}$ does not converge.

(b) For $\mathbf{A}$ nonsingular, the iterative scheme will converge provided what criteria is met?

(c) Find the first two iterations of the Jacobi method for the linear system

$$\begin{aligned} 3x_1 - x_2 + x_3 &= 1 \\ 3x_1 + 6x_2 + 2x_3 &= 0 \\ 3x_1 + 3x_2 + 7x_3 &= 4 \end{aligned}$$