Doctoral qualifying exam: Linear algebra and numerical analysis.

September 3, 1997.

You have three hours for this exam. Show all working in the books provided.

1. Suppose A is a symmetric 3×3 matrix with eigenvalues 0, 1 and 2.

(a) What are the properties of the unit eigenvectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ of the matrix A?
(b) Describe the four fundamental subspaces of A in terms of the unit eigenvectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.
(c) Find a vector $\mathbf{x}$ that satisfies the equation $A\mathbf{x} = \mathbf{v} + \mathbf{w}$. Is $\mathbf{x}$ unique?
(d) What conditions must the vector $\mathbf{b}$ satisfy for the equation $A\mathbf{x} = \mathbf{b}$ to have a solution?
(e) If the unit eigenvectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ form the columns of a matrix S, what is S^{-1}?

2. (a) If A is a $m \times n$ matrix and B is a $n \times m$ matrix with $n < m$, show that AB is singular.

(b) If A is a $n \times n$ matrix and $\mathbf{b}$ is a $n \times 1$ vector, show that only one of the following two systems of equations is consistent:

$$\text{(i) } A\mathbf{x} = \mathbf{b} \qquad\qquad \text{(ii) } A^T\mathbf{y} = \mathbf{0},\ \mathbf{b}^T\mathbf{y} = 1.$$

3. For a given a matrix A, find two different square matrices R_1 and R_2 such that $A = R_1^T R_1 = R_2^T R_2$.

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}$$

4. (a) Show that the equation $x = \frac{2}{3}x + \frac{1}{x^2}$ has a real solution α.

(b) Find the largest interval $[a, b]$ containing α such that for every $x_0 \in [a, b]$ the iteration scheme $x_{n+1} = \frac{2}{3}x_n + \frac{1}{x_n^2}$, $n = 0, 1 \ldots$, converges to α.

(c) Give the order of convergence of this iteration method; if the convergence is linear give the rate of linear convergence.

5. Let $I = \int_1^4 (x-1)^2 dx$.

(a) Using trapezoidal rule integration with four equally spaced nodes, estimate the value of I. Repeat this calculation using seven equally spaced nodes.

(b) Deduce the order of convergence of this method using the exact solution and your answer from part (a).

(c) Use Richardson extrapolation to obtain a higher order estimate of I using your answers from part (a). What is significant about this higher order estimate? Is this what you would expect - explain why?

6. (a) Consider the initial value problem

$$y' = x - x^3, \quad y(0) = 0$$

Suppose we use the (forward) Euler method with stepsize h to compute approximate values y_j for $y(x_j)$, where $x_j = jh$. Find an explicit formula for y_j and the error $e(x_j) = y_j - y(x_j)$, and show that the error, for x fixed, tends to zero as $h = x/n \to 0$.
(**Hint:** you may need to use one or more of the formulas $\sum_{k=1}^n k = n(n+1)/2$, $\sum_{k=1}^n k^2 = n(n+1)(2n+1)/6$, $\sum_{k=1}^n k^3 = n^2(n+1)^2/4$.)

(b) The modified Euler method

$$y_{j+1} = y_j + hf\left(x_j + \frac{h}{2}, y_j + \frac{h}{2} f(x_j, y_j)\right)$$

is often used to obtain a numerical solution to $y' = f(x,y)$. Show that this method is second order accurate, i.e., the error at the end of each step tends to zero like h^2.