LINEAR ALGEBRA / NUMERICAL METHODS QUALIFYING EXAMINATION Fall 1996

Problem #1

Let A and B be complex, $n \times n$, positive definite matrices. Prove the following:

(A). $A + B$ is positive definite.

(B). If $c > 0$, then cA is positive definite.

(C). A^{-1} exists and is positive definite.

Problem #2

Let A be a complex, 3×3 matrix. Describe all possible types of Jordan canonical forms for A and their corresponding characteristic polynomials.

Problem #3

Consider the system,

$$\lambda x = Ax + y,$$

where x and y are real n-vectors, A is a real $n \times n$ matrix and $\lambda > 0$ is larger than the absolute values of all the eigenvalues of A.

(A). Describe how a unique solution x can be found for this equation by iteration.

(B). Find an estimate for the error after three iterations.

Problem #4

Let $I = \int_0^3 x^2 \, dx$.

(A). Estimate the value of I using trapezoidal rule integration.

(B). Find the order of this integration method.

(C). Use Richardson extrapolation to obtain a higher order estimate of I from the estimate in part (A). Why is this higher order estimate exact?

Problem #5

(A). Give Euler's method for the following system of ordinary differential equations:

$$\begin{aligned} dx/dt &= f(x,y) \\ dy/dt &= g(x,y). \end{aligned}$$

(B). For the case where $f(x,y) = y$ and $g(x,y) = -x$, find a matrix A such that your iteration formula can be written as

$$\begin{bmatrix} x_{i+1} \\ y_{i+1} \end{bmatrix} = A \begin{bmatrix} x_i \\ y_i \end{bmatrix}.$$

(C). Find another matrix $\tilde{A}$ such that this last formula becomes *exact* when A is replaced by $\tilde{A}$.

Problem #6

The steady-state temperature distribution in the interior of a cylinder of radius 1 is described by the solution $y(x,\alpha)$ of the following nonlinear boundary value problem:

$$\begin{aligned} \frac{d^2y}{dx^2} &= -\frac{1}{x}\frac{dy}{dx} - \alpha e^y \\ \frac{dy(0)}{dx} &= 0 \\ y(1) &= 0 \end{aligned}$$

The parameter α is defined as

$$\alpha = \frac{\text{heat generation}}{\text{conductivity}}, \qquad 0 < \alpha \leq 0.8.$$

Although the right-hand side of the above differential equation is singular at $x = 0$, there exists a solution $y(x,\alpha)$ which is analytic for small $|x|$. In spite of this, a numerical solution to this equation may fail near $x = 0$.

(A). Explain why a numerical solution to this equation may fail near $x = 0$.

(B). Modify this problem so that it can be solved numerically for $0 \leq x \leq 1$. (Hint: Use Taylor series).

(C). Show how to obtain a numerical solution to this modified problem.