Doctoral Qualifying Exam in Applied Mathematics

Part B: Algebra

September 18, 1995

Problem 1.

(a) Let $T : \mathcal{R}^{\ni} \to \mathcal{R}^{\ni}$ be defined as $T(x, y, z) := (x + y + z, y - 2z, x + 2y - z)$. Show that $\mathcal{R}^{\ni}$ is linearly isomorphic with $\ker T \oplus imT$, where $\ker T := \{(x, y, z) : T(x, y, z) = 0\}$ and $imT := \{T(x, y, z) : (x, y, z) \in \mathcal{R}^{\ni}\}$.

(b) Prove or disprove the following generalization of (a): If $T : X \to Y$ is a linear map between finite -dimensional vector spaces X and Y, then X is linearly isomorphic with $\ker T \oplus imT$.

Problem 2.

The solutions of the linear system

$$\begin{aligned} x_1 + 3x_2 - 2x_3 &= 1 \\ x_1 + x_2 - 2kx_3 &= -3 \\ -x_1 + (k-3)x_2 + 2kx_3 &= k \end{aligned}$$

depend on the parameter k. Find the values of k for which the system has a unique solution, an infinity of solutions, and no solution. In the first two cases give the general solution or its general form.

Problem 3.

(a) Prove the Cayley-Hamilton theorem for diagonalizable matrices. That is, prove that if a matrix A is diagonalizable, it satisfies its characteristic equation.

(b) Use (a) to show that if a diagonalizable matrix A has characteristic equation $\lambda^2 + \frac{1}{2}\lambda = 0$, then $(I - A)^{-1} = I + \frac{2}{3}A$.

Problem 4.

Use an orthogonal transformation to diagonalize the quadratic form in

$$4(x^2 + y^2 + z^2 + xy + xz + yz) = 1$$

and use the result to describe the quadric surface.

Problem 5.

(a) Show that if A is any $p \times q$ matrix, then $A^H A$ is hermitian and its eigenvalues are real and

non-negative.

(b) Given that any $p \times q$ matrix has a decomposition $A = U \Sigma V^H$, where U and V are $p \times p$

and $q \times q$ unitary matrices, respectively, and Σ is a $p \times q$ matrix with $\Sigma_{ij} = 0$ for $i \neq j$ and

$\Sigma_{ii} = \sigma_i$ real and non-negative, show that: the eigenvalues of $A^H A$ are the columns of V and

the associated eigenvalues are the σ_i^2. Also show that if u_i and v_i are the columns of U and

V, respectively, then $Av_i = \sigma_i u_i$ for $1 \leq i \leq \min\{p, q\}$.

Problem 6.

Prove that every group of order 9 is abelian.

Problem 7.

Let S be the pattern of a planar square lattice (i.e., graph paper).

(a) Determine the symmetry group of S; that is, the crystallographic group that leaves S

invariant.

(b) Compute all finite subgroups of the symmetry group in (a).

(c) Which, if any, of the finite subgroups in (b) are normal?

Problem 8.

Prove, using Galois theory or any other method, that "triangulating the square" can be accomplished using classical straightedge and compass construction. That is, prove that one can construct an equilateral triangle whose area is equal to that of a given square using only a straightedge and compass.