Doctoral Qualifying Exam in Applied Mathematics
Part B: Algebra. September 13, 1994

Problem 1. (a) Let A be hermitian: prove all eigenvalues are real.

(b) Prove or disprove: if A is normal all of its eigenvalues are real.

Problem 2. Consider the linear differential equation in $\mathcal{R}^3$

$$\dot{x} = Ax$$

where $\dot{x} \equiv \frac{dx}{dt}$ and

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

(a) Show that the set M of initial points for which the trajectories converge to the origin exponentially with increasing t is a two-dimensional subspace of $\mathcal{R}^3$.
(b) Find M.

Problem 3. Taking into account that numerical methods for analysis usually result in small errors in numerical values due to approximation and round-off errors, explain why it is unwise to use the Jordan canonical form of a matrix when employing numerical methods. Illustrate your explanation with examples where appropriate.

Problem 4. Classify (up to group isomorphism) all groups of order 4. Prove all steps in the classification.

Problem 5. Consider the infinite pattern of equally spaced vees in the plane shown below

$$\cdots V\ V\ V\ V\ V \cdots$$

(a) Find a set of generators for the subgroup G of isometries of $\mathcal{R}^2$ which have the pattern invariant.

(b) Prove that the subgroup T of translations in G is a normal subgroup and that G/T is cyclic.

Problem 6. Compute the Galois group G of $x^4 - 5x^2 + 6$ over the rationals $\mathcal{Q}$. Determine all the subgroups of G and the corresponding subfields of the splitting field.