Doctoral Qualifying Exam, Linear Algebra and Numerical Methods.

August 30, 2000.

Problem 1. (a) Suppose

$$A = \begin{pmatrix} 100 & 7 & 5 & 3 \\ 9 & 2 & 2 & 1 \\ 8 & 7 & 4 & 2 \\ 600 & 2 & 5 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 600 & 2 & 5 & 1 \\ 9 & 2 & 2 & 1 \\ 8 & 7 & 4 & 2 \\ 100 & 7 & 5 & 3 \end{pmatrix}.$$

Find a 4×4 matrix S such that $SA = B$.

(b) Determine the rank of each of the following matrices:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \qquad B = \begin{pmatrix} 1 & 2 & 4 & \pi & 0 \\ 0 & 0 & 1 & 0 & 7 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad C = \begin{pmatrix} 5 & 1 & 1 & 1 & 1 \\ 1 & 5 & 1 & 1 & 1 \\ 1 & 1 & 5 & 1 & 1 \\ 1 & 1 & 1 & 5 & 1 \\ 1 & 1 & 1 & 1 & 6 \end{pmatrix}.$$

Justify your answers.

Problem 2. Let A be a Hermitian matrix.

(a) Show that all eigenvalues of A are real.

(b) Show that eigenvectors of A corresponding to different eigenvalues are orthogonal.

(c) Schur's lemma states that given any square matrix A we can find a unitary matrix U such that $T = U^H A U$ is upper triangular. Explain how Schur's lemma guarantees that all Hermitian matrices are diagonalizable.

Problem 3. Compute

$$\lim_{N \to \infty} \begin{pmatrix} 1 & \frac{\pi}{2N} \\ -\frac{\pi}{2N} & 1 \end{pmatrix}^N.$$

Problem 4.

1. Given the iteration $x_{n+1} = g(x_n)$, find the fixed point(s) and determine their stability when $g(x) = 1 + x - x^2/4$.

2. Given the iteration $x_{n+1} = g(x_n)$, find the fixed point(s) and determine their stability when $g(x) = 2(x-1)^{1/2}$, $x \geq 1$.

3. Prove the following two theorems about fixed points:
(a) Assume that $g \in C[a,b]$. If the range of the mapping $y = g(x)$ satisfies $a \leq y \leq b$ for all $a \leq x \leq b$, then g has a fixed point in $[a,b]$.
(b) Further, suppose that $g'(x)$ is defined on (a,b) and there is a positive constant $K < 1$ such that

$$|g'(x)| \leq K < 1, \text{ for all } x \in (a,b),$$

then g has a unique fixed point P in $[a,b]$.

Problem 5. Consider the system of two differential equations

$$\begin{aligned} x' &= \alpha x + \beta y \qquad & x(0) = 2 \\ y' &= \beta x + \alpha y \qquad & y(0) = 0 \end{aligned}$$

1. Find the analytic solution to this system.

2. Determine the approximate numerical solution at the n^{th} step for Euler's method.

3. Assuming $\alpha < \beta < 0$, what restriction on the time step is needed so that the numerical solution is bounded.

Problem 6. Derive a numerical differentiation formula of order $O(h^4)$ by applying Richardson extrapolation to

$$f'(x) = \frac{1}{2h}[f(x+h) - f(x-h)] - \frac{h^2}{3!}f^{(3)}(x) - \frac{h^4}{5!}f^{(5)}(x) - \ldots.$$

Give the error term of order $O(h^4)$.