Ph.D Qualifying Exam in Linar Algebra and Numerical Analysis

August 26, 2002

Problem 1.

(a) Show that the intersection of two subspaces is a subspace.

(b) Let S be the span of $\{x, y, z\}$ where $x = (1, 0, 1, 1)$, $y = (1, -1, 0, 0)$ and $z = (1, 0, -1, 0)$. Let T be the span of $\{u, v, w\}$ where $u = (1, 0, 0, -1)$, $v = (0, 0, 1, -1)$ and $w = (1, 1, 1, 0)$. Find a basis for the intersection $S \cap T$.

Problem 2.

(a) Prove that the eigenvalues of a real symmetric matrix are real.

(b) Fully characterize the eigenvalues of an $n \times n$ real skew-symmetric matrix ($A = -A^t$). A full characterization is a description of the set of eigenvalues such that (i) the eigenvalues of every $n \times n$ real skew-symmetric matrix satisfy the description and (ii) for every set of numbers satisfying the description there is an $n \times n$ real skew-symmetric matrix having those numbers as its eigenvalues.

(c) What are the possible values for the determinant of a (real) orthogonal matrix?

(d) Can a normal matrix be singular? (Prove your answer.)

Problem 3. Let A be a 2×2 matrix with complex entries. Consider a sequence of 2×2 matrices $\{X_n\}_{n=0,1,\ldots}$ satisfying

$$X_{n+1} \;=\; AX_n \;-\; X_n A.$$

Note that the mapping relating X_n to X_{n+1} is linear.

(a) Show that $tr(X_n) \to 0$ as $n \to \infty$ where $tr(X)$ is the "trace of X."

(b) Suppose that the distance between any two eigenvalues of A is less than one. Show that $X_n \to 0$ as $n \to \infty$.

Problem 4. Consider the ordinary differential equation

$$y'' + \omega^2 y = 0,$$

which has solutions of the form

$$y(t) = e^{\pm i\omega t}.$$

The equation is to be solved numerically using the following scheme:

$$\frac{y^{n+1} - 2y^n + y^{n-1}}{\Delta t^2} + \omega^2 \frac{y^{n+1} + y^{n-1}}{2} = 0. \tag{1}$$

(a) Determine the order of accuracy of the numerical scheme.

(b) Look for a solution of the scheme (1) of the form $y^n = e^{i\tilde{\omega} t_n}$, where $t_n = n\Delta t$, and determine $\tilde{\omega}$. What is the significance of $\tilde{\omega}$ as compared with ω?

(c) Quantify this by determining the leading order of the Δt-small expansion of $|\omega - \tilde{\omega}|$. Comment on your results and in particular on the appropriateness of the scheme (1).

Problem 5. Solve each of the following.

(a) Set up the Newton iterations for the function $f : (0, \infty) \to \mathbf{R}$ given by

$$f(x) = \frac{\ln 2}{\pi} \sin\left(2\pi \frac{\ln x}{\ln 2}\right) + 1.$$

Show that the Newton iterations starting with $x_0 = 1$ converge, but that the limit is not a zero of f. Explain your findings.

(b) Show that

$$\lim_{n\to\infty} \sqrt{2 + \sqrt{2 + \sqrt{2 + \ldots + \sqrt{2}}}} = 2.$$

(c) To find a root of $f(x) = 0$ by iteration, re-write the equation as

$$x = x + cf(x) \equiv g(x)$$

for some constant $c \neq 0$. If α is a root of $f(x)$ and if $f'(\alpha) \neq 0$, how should c be chosen in order that the sequence $x_{n+1} = g(x_n)$ converges to α?

Problem 6. Consider the boundary value problem

$$-u''(x) = f(x), \qquad x \in [0, 1],$$

$$u(0) = u(1) = 0,$$

with $f(x)$ a given continuous real function.

(a) Choosing an equidistant subdivision of the interval $[0, 1]$, into $(n + 1)$ intervals of size $h = 1/(n + 1)$, and approximating the derivative by second order differences, cast the boundary value problem into the matrix problem

$$A\mathbf{x} = \mathbf{b}.$$

State exactly what A, $\mathbf{x}$ and $\mathbf{b}$ are.

(b) Construct the Jacobi iteration for the solution of (b) above, in the form

$$\mathbf{x}_{k+1} = \mathcal{L}_1 \mathbf{x}_k + \mathcal{L}_2 \mathbf{b},$$

and give the matrices $\mathcal{L}_1$ and $\mathcal{L}_2$. Choose your scheme so that the matrix $\mathcal{L}_1$ has positive entries.

(c) Verify that $\mathcal{L}_1$ has eigenvalues

$$\lambda_j = \cos \frac{\pi j}{n+1}, \quad j = 1, \ldots, n,$$

and associated eigenvectors $\mathbf{v}_j$ with components

$$v_{j,m} = \sin \frac{\pi j m}{n+1}, \quad m = 1, \ldots, n, \quad j = 1, \ldots, n.$$

Hence, show that the spectral radius of the Jacobi method for this problem is approximately given by $1 - \frac{1}{2}\pi^2 h^2$ for small h. What can you deduce about the convergence of different discretizations?

[Hint: For the eigenvalue/eigenvector calculation you can quote the result $\frac{1}{2} \sin \frac{\pi j (m-1)}{n+1} + \frac{1}{2} \sin \frac{\pi j (m+1)}{n+1} = \cos \frac{\pi j}{n+1} \sin \frac{\pi j m}{n+1}$.]