Ph.D Qualifying Exam in Linear Algebra and Numerical Methods

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Problem 1. Consider the linear system

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \gamma \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix},$$

where α, β and γ are scalar constants.

(a) Find a basis for the left nullspace of the matrix on the left hand side of this system.

(b) Find necessary and sufficient conditions on α, β, γ for this system to have a solution.

(c) Find the general solution of the system.

Problem 2. In this problem, N is a positive integer; I is the $N \times N$ identity matrix; A, B and C are real $N \times N$ matrices; and $\hat{v}$ is a real N component vector of unit length.

(a) Show $A = I - 2\hat{v}\hat{v}^t$ satisfies $A^t A = I$ and $A^2 = I$.

(b) What are the eigenvalues of A?

(c) Suppose $B^t B = I$. What are the possible eigenvalues of B? That is, how does the condition $B^t B = I$ constrain the eigenvalues of B?

(d) Suppose $C^2 = I$. What are the possible eigenvalues of C? That is, how does the condition $C^2 = I$ constrain the eigenvalues of C?

(e) Suppose $(I + D)^2 = I$ and all the eigenvalues of D have magnitude less than two. Show $D = 0$.

Problem 3. Show that for large integer values of N

$$\begin{bmatrix} 1 & \frac{2\pi}{N} \\ -\frac{2\pi}{N} & 1 \end{bmatrix}^N = \left(1 + \frac{2\pi^2}{N}\right) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + O(N^{-2}).$$

You may find the Taylor series for e^z and $\ln(1+z)$ helpful:

$$\begin{aligned} e^z &= \sum_{n=0}^{\infty} \frac{1}{n!} z^n = 1 + z + \frac{1}{2} z^2 + \ldots \\ \ln(1+z) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^n = z - \frac{1}{2} z^2 + \frac{1}{3} z^3 + \ldots \end{aligned}$$

Problem 4. Consider the IVP

$$u'(t) = 0, \quad t \in [0, T], \quad u(0) = U_I.$$

The solution to this problem is $u(t) = U_I$. The two step scheme

$$u_{n+2} - 4u_{n+1} + 3u_n = -2hf_n$$

for this equation is consistent with order p.

1. Find p.

3. Determine if the scheme is convergent.

Problem 5. An iterative scheme producing sequences $\{x^m\}$ of approximations to x^* **converges superlinearly** if there exists a sequence $\{C_m\}$ of nonnegative constants such that $C_m \to 0$ and

$$\left|x^* - x^{(m+1)}\right| \leq C_m \left|x^* - x^m\right|.$$

1. **Prove the following:** Suppose that the iteration function Φ has a fixed point x^* and that Φ is a contraction on some neighborhood $(x^* - \delta, x^* + \delta)$ of x^*, where $\delta > 0$. Let $x^{(0)}$ be any initial guess lying in $(x^* - \delta, x^* + \delta)$, and let $\{x^{(m)}\}$ be the sequence of iterates generated by the successive substitution into the scheme

   $$x^{(m+1)} \leftarrow \Phi\left(x^{(m)}\right).$$

   Then each $x^{(m)} \in (x^* - \delta, x^* + \delta)$, and $x^{(m)} \to x^*$ as $m \to \infty$.

2. Show that a sequence $\{p_n\}$ that converges to p with order $\alpha > 1$ converges superlinearly to p.

3. Show that $p_n = \frac{1}{n^n}$ converges superlinearly to zero but is not convergent of order α for $\alpha > 1$.

Problem 6. Consider the nonlinear equation $f(x) = 0$.

1. State an algorithm that approximates the root of the equation using the Bisection method.
2. Derive the rate and order of convergence of the Bisection method.
3. State an algorithm that approximates the root of the equation using Newton's method.
4. Derive the rate and order of convergence of Newton's method.
5. Discuss the convergence for $f(x) = \sqrt[3]{x} = 0$.