Doctoral qualifying exam: Linear algebra and numerical analysis.

January 21, 1999.

You have three hours for this exam. Show all working in the books provided.

1. (a) Are the matrices A and B given below diagonalizable? Explain your answer. Find the Jordan form of the matrices A and B where

$$A = \begin{pmatrix} 4 & -2 & 1 \\ 2 & 0 & 1 \\ 2 & -2 & 3 \end{pmatrix},$$

and

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{pmatrix}.$$

(b) How are the pivots and the eigenvalues of a symmetric matrix related? What conclusions can you draw about the eigenvalues of the matrix A from the pivots of $A - 3I$ when

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

2. (a) Prove that if $A = A^H$, then for all complex vectors x the number $x^H Ax$ is real.
(b) In the space of polynomials P_2, let $v_1 = 2t^2 + t + 2$, $v_2 = t^2 - 2t$, $v_3 = 5t^2 - 5t + 2$, and $v_4 = -t^2 - 3t - 2$. Determine whether or not the vector $u = t^2 + t + 2$ belongs to the space spanned by $v_1,\ v_2\ , v_3,\ v_4$.

3. (a) Prove that if $S = \{v_1, v_2, \ldots, v_n\}$ is a basis for a vector space V and $T = \{w_1, w_2, \ldots, w_r\}$ is a linearly independent set of vectors in V, then $r \leq n$.
(b) Let $S = \{q_1, q_2, \ldots, q_n\}$ be an orthonormal basis for an inner product space V. Prove that any vector u in V is expressed in terms of the basis vectors in the following way: $u = q_1^T uq_1 + q_2^T uq_2 + \ldots + q_n^T uq_n$.

4. Show that the one-step method given by

$$\begin{aligned} \Phi(x, y; h) &= \frac{1}{6}\{k_1 + 4k_2 + k_3\} \\ k_1 &= f(x, y) \\ k_2 &= f(x + \frac{h}{2}, y_n + \frac{h}{2}k_1) \\ k_3 &= f(x + h, y_n + h(-k_1 + 2k_2)) \end{aligned}$$

(called the 'simple Kutta formula') is of third order.

5. (a) Determine whether or not the following modified Euler's method is stable:

$$\begin{aligned} y_0 &= \alpha \\ y_{n+1} &= y_n + \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_n + hf(t_n, y_n)) \end{aligned}$$

(b) Determine whether or not the following multistep method is stable:

$$y_{n+1} = -\frac{3}{2}y_n + 3y_{n-1} - \frac{1}{2}y_{n-2} + 3hf(t_n, y_n)$$

6. Use the fixed point theorem to show that the sequence

$$x_n = \frac{1}{2}x_{n-1} + \frac{1}{x_{n-1}}$$

converges for any $x_0 > 0$.