Doctoral Qualifying Exam:
Linear Algebra and Numerical Analysis.

Thursday, January 22, 1998.

You have three hours for this exam. Show all working in the books provided.

1. (a) Suppose that the matrices A and B are related by the similarity transformation $B = M^{-1}AM$. Show the relation between the eigenvalues and eigenvectors of A and B.
(b) Show that all eigenvalues λ of a unitary matrix U have absolute value $|\lambda| = 1$.
(c) Show that for a unitary matrix U, eigenvectors corresponding to different eigenvalues are orthogonal.

2. (a) For the matrix A below, find an orthonormal set of vectors q_1, q_2, q_3 for which q_1 and q_2 span its column space. Which fundamental subspace of A contains q_3?

$$A = \begin{pmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{pmatrix}.$$

(b) Suppose P is a projection onto the subspace S and Q is a projection onto the orthogonal complement of S. What are $P + Q$ and PQ?

3. (a) Prove that if V and W are three-dimensional subspaces in R^5, then V and W must have a non-zero vector in common.
(b) Suppose that the only solution of a system $A\mathbf{x} = \mathbf{0}$ is $\mathbf{x} = \mathbf{0}$ where A is $m \times n$. What is the rank of A and why?

4. Consider the following system of ordinary differential equations, which occurs in population growth dynamics

$$\frac{dx}{dt} = (b - d)x \qquad \frac{dy}{dt} = (b - d)y + rb(x - y)$$

where b, d, and r are constants and x, y are time-dependent population densities.
(a) By introducing the variable $p = y/x$, show that these equations can be combined into the single equation

$$\frac{dp}{dt} = rb(1 - p)$$

(b) Write down the difference equation for the approximate solution by Euler's method. Solve this difference equation for $p_n = p(n\Delta t)$ exactly.

(c) Solve the differential equation for p exactly. Show that for t fixed, the solution in (b) approaches the exact solution as $\Delta t = t/n \to 0$.

5. Give a numerical procedure to calculate $x = 1/a$ for any given $a \neq 0$ *without* using division. For which starting values x_0 will your method converge?

6. Given the matrix

$$A = \begin{pmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & -1 & 0 & 2 \end{pmatrix},$$

show that Jacobi's method for solving $Ax = b$ does not converge. (Recall that Jacobi iteration takes the form $x_{n+1} = Tx_n + c$ where $T = -D^{-1}(L+U)$ and $c = D^{-1}b$ with $A = D + L + U$ and D diagonal, L lower diagonal, and U upper diagonal.