LINEAR ALGEBRA / NUMERICAL METHODS QUALIFYING EXAMINATION
February 4, 1997

Problem #1

Find the orthogonal projection of the vector $\mathbf{u}$ on the nullspace of matrix A, where

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{pmatrix}$$

and

$$\mathbf{u} = \begin{pmatrix} 5 \\ 6 \\ 7 \\ 2 \end{pmatrix}.$$

Problem #2

(A). Let A be an $n \times n$ matrix and let λ_1 be an eigenvalue of multiplicity m. If A is diagonalizable what is the rank of $(A - \lambda_1\, I)$?

(B). Let A be an $n \times n$ matrix with an eigenvalue λ_1 of multiplicity n. Show that A is diagonalizable if and only if $A = \lambda_1\, I$.

Problem #3

(A). For any square matrix A, show how the determinant of A^H is related to the determinant of A (A^H is the conjugate transpose of A).

(B). Prove that the determinant of any Hermitian matrix is real.

(C). Factor the following matrix A in the form CC^H.

$$A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{pmatrix}.$$

Problem #4

(A). Set up Newton's method for systems of equations to find an approximate solution to the following equations:

$$\begin{aligned} 3x_1^2 &= x_2^2 \\ 3x_1x_2^2 &= x_1^3 + 1 \end{aligned}$$

Use $\vec{x} = (1,1)^t$ as your initial approximation and find the next approximation.

(B). A weakness of Newton's method for solving systems of nonlinear equations is that an accurate initial approximation to the solution is needed to ensure convergence. What other method could you use if you were given a poor initial approximation?

Problem #5

Consider the predictor method given by

$$y_{j+2} + a_1 y_{j+1} + a_0 y_j = h[b_0 f(x_j, y_j) + b_1 f(x_{j+1}, y_{j+1})].$$

(A). Determine a_0, b_0, and b_1 as a function of a_1 such that the method has order at least 2.

(B). For which values of a_1 is the method stable?

Problem #6

Consider the boundary value problem

$$y'' - p(x)y' - q(x)y = r(x), \quad y(a) = \alpha, \quad y(b) = \beta$$

with $q(x) > 0$ for $a \leq x \leq b$. We are seeking approximate values u_i for the exact values $y(x_i)$, $i = 1, 2, ..., n$, at the points $x_i = a + ih$, $h = (b-a)/(n+1)$. By replacing $y'(x_i)$ and $y''(x_i)$ by their appropriate second order difference formulae for $i = 1, ..., n$, and putting $u_0 = \alpha$, $u_{n+1} = \beta$, one obtains the following system of equations for the vector $u = [u_1, u_2, ..., u_n]^T$

$$Au = k \quad A \text{ an } n \times n \text{ matrix}, \; k \text{ a vector.}$$

(A). Determine A and k.

(B). Show that the system of equations is uniquely solvable for h sufficiently small.