Qualifying Exam:
Linear Algebra/Numerical Methods
Spring 1996

1. Roughly how many iterations are required for $x_{n+1} = \sin x_n, x_0 = 1$ to converge to within 10^{-6} of the true root at $x = 0$?

2. Show that no square matrices A and B can satisfy $AB - BA = I$.

3. Evaluate

$$\begin{vmatrix} 1+a_1 & a_2 & a_3 & \cdots & a_n \\ a_1 & 1+a_2 & a_3 & \cdots & a_n \\ a_1 & a_2 & 1+a_3 & \cdots & a_n \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_1 & a_2 & a_3 & \cdots & 1+a_n \end{vmatrix} \tag{1}$$

4. Consider $f(x) = x^3 - 1$.

   - Write down and simplify the Newton-Raphson iteration.
   - A colleague tries the iteration with the following starting guesses: $10, -10, -1000$ but always get the same root. Which root does he get? Why? Suggest a starting guess that might give the other roots.
   - Draw a picture with several tangents to illustrate how Newton's method is working for this problem. What will happen if we start with an initial guess $x_0 = 0$?
   - Strangely, your colleague is insistent about his choice of starting guesses but will consider a different root finder. Which of the following could give a different root than what he's been getting.
     - secant method
     - regula falsi
     - Muller's method

5. Consider the following boundary value problem:

$$u'' + f(x)u = g(x) \tag{2}$$
$$u(a) = \alpha \tag{3}$$
$$u'(b) = \beta \tag{4}$$

   H.B. Keller has suggested the following approach to discretizing it. Define a finite-difference mesh $x_i = a + ih, i = 0, \ldots, N+1$ where $h = \frac{b-a}{N+.5}$. Thus $x_0 = a$ but the point b is straddled by x_N and x_{N+1}.

   - Using Keller's mesh with a centered three-point finite-difference approximation to the second derivative and a centered two-point approximation to the first derivative, give a discretized equivalent to our original BVP. Write your result in matrix form for the unknowns $x_i, i = 0, \ldots, N$. (We are not interested in u_{N+1}.)
   - Discuss the pros and cons of this method vs. the standard approach where the endpoints are part of the grid.

- How will the run-time vary as a function of N?

6. A certain quadrature scheme yields an estimate of the integral I with the following dependence on mesh width h,

$$I = a + bh^4 + ch^5 + dh^6 + \dots \tag{5}$$

We wish to use Richardson extrapolation to improve the accuracy.

- What is the appropriate two-point extrapolation formula?
- What is the appropriate three-point extrapolation formula?

7. Consider the matrix

$$A = \begin{bmatrix} 101/2 & -99/2 \\ -99/2 & 101/2 \end{bmatrix} \tag{6}$$

In exact arithmetic, roughly how many iterations would be needed to obtain the eigenvector corresponding to the largest eigenvalue with 3 significant digits, using the following starting guesses in the power method:

- (1, 1)
- (1, -1)
- (1, 0)

8. Consider

$$u'' + \lambda^2 u = 0 \tag{7}$$
$$u(0) + u'(0) = 0 \tag{8}$$
$$u(1) + u'(1) = 0 \tag{9}$$

The exact eigenvalues are $\lambda_k = k\pi$. If we solve this problem using finite differences with the standard formula, we will get approximations to these eigenvalues $\beta_k(N) = 2N \sin \frac{k\pi}{2N}$ where N is the number of points in the mesh and $k = 1, \dots, N-1$

(a) Calculate $\beta_1(10)$, $\beta_1(20)$, $\beta_1(40)$.

(b) Build up the Richardson extrapolation table.

(c) Roughly, how large would N have to be to obtain the most accurate extrapolated value in your table by simple mesh refinement?

(d) How much slower would the mesh refinement be? (Solving the finite difference equations for $\beta(N)$ requires roughly $20N$ operations.)

9. Consider the matrices

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 1 & 1 \\ -1 & -1 & 2 \end{bmatrix} \tag{10}$$

Considering iterative methods for linear systems with A and B as coefficient matrices, would Jacobi converge? Gauss-Seidel? (Do not exchange the order of the equations.)