Doctoral Qualifying Examination
in
Applied Mathematics

Part B: Algebra

January 24, 1995

Problem 1. For the linear system

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & 1 \\ x_1 & + & kx_2 & + & 3x_3 & = & -1 \\ -x_1 & - & x_2 & + & (k-3)x_3 & = & 0 \end{array}$$

find values of k for which the system has a unique solution, an infinite number of solutions and no solutions. In the first two cases, give the solution or its general form.

Problem 2. Prove that the rank of the product AB of two $n \times n$ matrices A and B does not exceed the rank of A.

Problem 3. Use the singular value decomposition, the QR-decomposition or any other method to solve the least squares problem

$$\begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \approx \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

Problem 4. If the $p \times p$ matrix A is normal and I is the identity matrix, derive the conditions for solvability of the system

$$(A - \lambda I)x \;=\; b$$

as the parameter λ varies. When a solution exists, give the solution or its general form in terms of the eigensystem (λ_i, x_i) of A. (You may quote the result that a normal matrix has a complete orthonormal set of eigenvectors.)

Problem 5. Consider the differential equation

$$\frac{dx}{dt} = Ax$$

where A is a real, 6×6 matrix with Jordan canonical form

$$J = P^{-1}AP = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Find the general solution and compute the entries of the fundamental matrix except for the factors P and P^{-1}.

Problem 6. Let G be a group with respect to $\cdot$ and a be an arbitrary element of G. Prove that $H = \{x \in G \ : \ x \cdot a = a \cdot x\}$ is a subgroup of G.

Problem 7.

(a) Compute the crystallographic group G of the infinite pattern

$$\cdots \text{H H H H} \cdots$$

That is compute the subgroup of planar isometries leaving the pattern invariant.

(b) Determine all finite cyclic subgroups of G.

(c) Find the infinite cyclic subgroup of G and prove that it is invariant.

Problem 8. Let $p(x) = a_0 + a_1 x + a_2 x^2 + x^3$ be a third degree polynomial with rational coefficients. If the Galois group of this polynomial over the rational numbers is trivial, show that all the roots of p must be rational numbers.