Ph.D Qualifying Exam in Linear Algebra and Numerical Methods

January 18, 2002. 2:00-5:00pm.

Problem 1. Let $\mathbf{R}^3$ be euclidean 3-space with the standard inner product $\langle \cdot, \cdot \rangle$ and orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation defined by $T(\mathbf{e}_1) = (1,1,-2), T(\mathbf{e}_2) = (1,1,1), T(\mathbf{e}_3) = (-1,-1,2)$.

(a) Compute the kernel (null space) and image (range) of T and conclude that T has no inverse.

(b) Find an orthonormal basis for $K^\perp$, the orthogonal complement of the kernel K, and show that T restricted to $K^\perp$ is invertible.

Problem 2. Let $\mathbf{C}^n$ be complex n-space with the usual (complex) inner product. Define $U(n)$ to be the space of complex $n \times n$ matrices A such that $A^*A = AA^* = I$, where A^* is the conjugate transpose of A. Matrices in $U(n)$ are said to be *unitary matrices.* A matrix A is *skew-Hermitian* if $A + A^* = 0$, and the space of skew-Hermitian matrices is often denoted by $u(n)$.

(a) Prove that $A \in u(n) \Longrightarrow e^A \in U(n)$.

(b) Show that $B \in U(n) \Longrightarrow$there exists $A \in u(n)$ such that $e^A = B$.

Problem 3. If A is a real $n \times n$ matrix such that all eigenvalues of A^tA are positive, where A^t is the transpose of A, and $\|A\|$ is the standard matrix norm defined as $\|A\| = \max\{\|A\mathbf{x}\| : \|\mathbf{x}\| = 1\}$, where $\|\mathbf{x}\|$ is the standard euclidean norm of $\mathbf{x} \in \mathbf{R}^n$, show using variational techniques or other means that

$$\sqrt{\mu_m} \le \|A\| \le \sqrt{\mu_M},$$

where μ_m and μ_M are, respectively, the minimum and maximum eigenvalues of A^tA. (Hint: A^tA is real-symmetric)

Problem 4.

(a) Find constants γ_0 and γ_1 such that the scheme

$$\frac{\gamma_0 f(x) \;+\; \gamma_1 f(x+h)}{h} \qquad (*)$$

is an approximation of $f'(x)$ of highest possible order. What is the order?

(b) Find constants α_0, α_1 and α_2 such that the scheme

$$\frac{\alpha_0 f(x) \;+\; \alpha_1 f(x+h) \;+\; \alpha_2 f(x+2h)}{h} \qquad (**)$$

is an approximation of $f'(x)$ of highest possible order. What is the order?

(c) Apply (*) and (**) to approximating $f'(0) = 0$ where $f(x) = x^{4/3}$. What is the order of the error in each case? Explain any apparent discrepancies with earlier results.

Problem 5. A certain first-order method (first-order local truncation error) for autonomous ODE's ($\dot{x} = f(x)$) has the form

$$x_{n+1} \;=\; \alpha x_n \;+\; \beta x_{n-1} \;+\; \gamma h f(x_{n-1})$$

where α, β and γ are constants and $h > 0$ is the time step.

(a) Find formulas for β and γ in terms of α.

(b) Find a necessary and sufficient condition on α for the method to be stable.

Problem 6. Suppose $F : \mathbf{R} \to \mathbf{R}$ is a smooth bijection with a positive derivative. There exists a smooth inverse function $G(x)$ satisfying

$$F(G(x)) \;=\; G(F(x)) \;=\; x \qquad \forall x \in \mathbf{R}.$$

Moreover, G satisfies the autonomous differential equation

$$\frac{dG}{dx}(x) \;=\; \frac{1}{F'(G(x))}. \tag{$*$}$$

If $G(0)$, the unique root of F, is known then G is fully determined by (*) and this initial datum.

Suppose it is desired to estimate $G(h)$ in terms of $G(0)$ under the assumption that F and F' are known and easily computable. A simple method is to apply a step of Euler's method to (*). This method gives the following approximation for $G(h)$:

$$A \;=\; G(0) \;+\; \frac{h}{F'(G(0))}.$$

A plausible method for improving this approximation is to use A as an initial guess for a step of Newton's method applied to $\phi(x) = F(x) - h$. Note that $G(h)$ is the unique root of $\phi(x)$. Call the approximation of $G(h)$ obtained from this Newton step B.

(a) Find a formula for B in terms of A, F, F' and h.

(b) Show that $A = G(h) - \epsilon$ where $\epsilon = \frac{1}{2}h^2 G''(0) + \frac{1}{6}h^3 G'''(\xi)$ and $\xi \in [0, h]$.

(c) Show that $B - G(h) = O(h^4)$. Of what order would the error be if the Newton step followed a step of Heun's method[1] (rather than a step of Euler's method)? Of what order would the error be if two Newton steps followed a step of Euler's method? Of what order would the error be if three Newton steps followed a step of Heun's method?

[1] Heun's method for $x' = f(x)$ is $x_{n+1} = x_n + \frac{h}{2} f(x_n) + \frac{h}{2} f\big(x_n + h f(x_n)\big)$.