# Ph.D Qualifying Exam in Linear Algebra and Numerical Methods

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**Problem 1**. Consider the matrix

$$A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ 0 & 4 & -6 & -1 \\ 2 & 0 & 4 & 1 \end{pmatrix}.$$

(a) Determine the range, null space, rank and nullity of $A$.

(b) Show that the nullity of $A$ can be reduced by making an arbitrarily small change in one of the entries of the matrix. Can the nullity be reduced to zero?

**Problem 2**. Let

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } C = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

(a) Find the minimum and maximum eigenvalues of $B$ by solving optimization (variational) problems.

(b) Compute the standard euclidean norm $\|C\|$ by solving an optimization (variational) problem.

**Problem 3**. Let the eigenvalues of $M$, a real $n \times n$ matrix, all be negative real numbers. Explain why every solution $\mathbf{x} = \mathbf{x}(t)$ of the differential equation

$$\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt} = M\mathbf{x}$$

satisfies $\mathbf{x}(t) \to 0$ as $t \to \infty$.

**Problem 4**. Consider the set of points $\{(-1,0),(0,1),(1,0)\}$.

(a) Find the line best fitting the data in the "least-squares" sense.

(b) Find the quadratic polynomial interpolating these points.

(c) Find the natural cubic spline interpolating these points.

**Problem 5**. Consider the finite-difference approximation

$$f''(x) \approx \frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{\Delta x^2}.$$

(a) Find the first nonzero term in a Taylor series expansion for the error. This term will have the form $C\Delta x^d$ where $C$ and $d$ are to be determined.

(b) Use Richardson extrapolation and the result of (a) to find a fourth-order finite-difference approximation for $f''(x)$.

**Problem 6.** Suppose that $F : \mathbf{R} \to \mathbf{R}$ is a smooth bijection with a positive derivative. There exists a smooth inverse function $G(x)$ satisfying

$$F\left(G(x)\right) = G\left(F(x)\right) = x \quad \forall x \in \mathbf{R}.$$

Moreover, $G$ satisfies the autonomous differential equation

$$\frac{dG}{dx}(x) = \frac{1}{F'\left(G(x)\right)}. \tag{1}$$

If $G(0)$, the unique root of $F$, is known then $G$ is fully determined by (1) and this initial datum.

Suppose that it is desired to estimate $G(x)$ at $N$ equally spaced points on the interval $[0, 1]$. A simple method to make this estimate is to solve (1) using Euler's method. That is, $G_0 = G(0)$ and

$$G_{n+1} = G_n + \frac{h}{F'\left(G_n\right)} \quad (n = 0, 1, ..., N-1) \tag{2}$$

where $h = 1/N$.

A natural approach to improve Euler's method is to use an iteration of Newton's method after each Euler step. This improved method is given by $G_0 = G(0)$ and

$$\gamma_{n+1} = G_n + \frac{h}{F'\left(G_n\right)} \tag{3}$$

$$G_{n+1} = G_n - \frac{F(\gamma_{n+1}) - (n+1)h}{F'(\gamma_{n+1})} \tag{4}$$

for $n = 0, 1, ..., N-1$.

(a) Determine $p$ such that $G(1) - G_N = O\left(h^p\right)$ where $G_N$ is computed using (2). Explain the basis for your determination.

(b) Determine $q$ such that $G(1) - G_N = O\left(h^q\right)$ where $G_N$ is computed using (3) and (4). Explain the basis for your determination.