PhD Qualfying Examination
Part B: Linear Algebra and Numerical Methods
January 15-16, 2004

1. Suppose H is a Hermitian matrix and U is a unitary matrix such that

$$U^*HU = \begin{bmatrix} 1.02 & -.03 & .01 \\ -.03 & .87 & -.01 \\ .01 & -.01 & 7.32 \end{bmatrix}$$

   Give estimates for the eigenvalues of H and theoretical bounds on the errors, and explain how these are obtained. Using the fact that the norm of a Hermitian matrix P satisfies $||P|| \leq \max_i \sum_{j=1}^{n} |p_{ij}|$, give numerical bounds on the errors.

2. Consider the matrix

$$A = \begin{bmatrix} -7 & 4 \\ -8 & 1 \end{bmatrix}$$

   Find the fundmental solution $X(t)$ $(t \geq 0)$ to the homogeneous differential equation

$$\frac{dx}{dt} = Ax$$

   Given an initial vector $x(0)$ and time-dependent vector $b(t)$, show how $X(t)$ can be used to find the solution to the inhomogeneous initial-value problem

$$\frac{dx}{dt} = Ax + b(t)$$

3. Let A be an $n \times n$ non-Hermitian (complex) matrix with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. First, show that A^* has eigenvalues $\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n$. Now, if u^j is an eigenvector of A belonging to λ_j and v^k is an eigenvector of A^* belonging to $\bar{\lambda}_k$, show that: $(u^j, v^k) = 0$ if $j \neq k$ and $(u^j, v^k) \neq 0$ if $j = k$.

4. This problem concerns the iteration

$$x_{n+1} = \frac{(6-2c)x_n^2 + (c-1)x_n}{6x_n - 1}$$

   where c is a constant.

   (a) Show that $\alpha = \frac{1}{2}$ is a fixed point of this iteration.

(b) Find the range of values of c for which this iteration will converge to the fixed point $\alpha = \frac{1}{2}$ assuming that x_0 is chosen sufficiently close to α.

(c) For what values of c, if any, will the order of convergence be greater than 1? Why?

5. Give an operation count for the solution of a linear system of n equations in n unknowns through the use of Gauss-Jordan elimination. Recall that in Gauss-Jordan elimination, elementary row operations are used to reduce the system of equations to diagonal form; then the resulting diagonal system is solved. Assume that neither pivoting nor scaling is used. Please keep the additions/subtractions counted separately from the multiplications/divisions. Recall that

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}, \qquad \sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}.$$

6. Consider the initial value problem

$$y' = f(x,y), \quad y(x_0) = y_0.$$

In order to numerically approximate the solution to this equation, the following method is used:

$$y_{n+1} = y_n + hF(x_n, y_n)$$

where h is the step size and

$$F(x,y) = \gamma_1 f(x,y) + \gamma_2 f(x + \alpha_1 h, y + \alpha_2 h f(x,y))$$

with $\gamma_1, \gamma_2, \alpha_1$, and α_2 parameters.

(a) Derive all possible order 2 schemes of the above form.

(b) In general, why is it impossible to derive an order 3 scheme of the above form?