Linear Algebra/Numerical Methods Qualifying Exam - June, 2004. Answer ALL questions. 3 hours.

1. Consider the space of real 4-vectors with the standard inner product and orthonormal basis $\{e_1, e_2, e_3, e_4\}$. Let L be a linear operator such that $L(e_1) = col(1,1,1,0)$, $L(e_2) = col(1,2,-1,0)$, $L(e_3) = col(-1,-2,1,0)$, and $L(e_4) = col(0,0,0,1)$.

   (a) Find the null space and range of L, and show that L is not invertible.

   (b) Find an orthonormal basis for the orthogonal complement of the null space of L. Show that L restricted to the orthogonal complement is invertible.

2. Let

$$M = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}, \quad K = \begin{bmatrix} 3 & 1 \\ 1 & 21 \end{bmatrix}.$$

   Solve the generalized eigenvalue problem $Kv^j = \lambda_j M v^j$ $(j = 1, 2)$, and find the matrix C such that $C^* M C = I$ and $C^* K C = diag(\lambda_1, \lambda_2)$. Given initial vectors $x(0)$ and $x'(0)$, show how the above results can be used to solve the initial-value problem:

$$Mx''(t) + Kx(t) = 0$$

3. Let V be the complex vector space spanned by the set of functions $\{1, \sin x, \cos x\}$, and let T be the linear operator on V defined by $T(f) = f'$, the derivative of f.

   (a) Find the eigenvalues of T and a complete set of eigenvectors or principal vectors, as appropriate.

   (b) Find a matrix S that puts T in the corresponding diagonal or Jordan canonical matrix form. Verify that S gives the desired result.

4. Consider the iteration

$$x_{n+1} = \frac{3x_n^4 + 30x_n^2 - 25}{8x_n^3}$$

   (a) Show that $\alpha = \sqrt{5}$ is a fixed point of this iteration.

   (b) Determine the order of convergence of this iteration to the fixed point $\alpha = \sqrt{5}$ assuming that x_0 is chosen sufficiently close to α.

5. Suppose the following values of a smooth function f: $\{f(x_0), f(x_1), f(x_2), f(x_3)\}$ are given. Assume that the node points $\{x_0, x_1, x_2, x_3\}$ are equispaced with node spacing h.

(a) Which of the following would you expect to be able to approximate more accurately using the given nodes and function values, $f'(x_0)$ or $f'(x_1)$? Why?

(b) Find the best possible approximations for $f'(x_0)$ and $f'(x_1)$ using the given nodes and function values.

(c) Find the error in each of the approximations from part **b** and compare with part **a**.

6. This problem involves the derivation of quadrature formulas through the use of interpolation.

(a) Find a polynomial $p(x)$ of degree ≤ 3 that interpolates a smooth function $f(x)$ with

$$\begin{aligned} p(x_0) &= f(x_0) = y_0, \quad p'(x_0) = f'(x_0) = y_0', \qquad (1) \\ p(x_1) &= f(x_1) = y_1, \quad p'(x_1) = f'(x_1) = y_1', \qquad (2) \end{aligned}$$

where the nodes $\{x_0, x_1\}$ are distinct and the values $\{y_0, y_0', y_1, y_1'\}$ are given. Write $p(x)$ in the form

$$p(x) = y_0 g_0(x) + y_0' g_1(x) + y_1 g_2(x) + y_1' g_3(x).$$

(b) What is the error in the using $p(x)$ to approximate $f(x)$?

(c) Assuming that $x_1 - x_0 = h$, derive a quadrature rule using the approximation

$$\int_{x_0}^{x_1} f(x)dx \approx \int_{x_0}^{x_1} p(x)dx.$$

(d) Derive an error formula for the quadrature rule in part **c**.

(e) Use the results from part **c** to obtain a composite quadrature rule for

$$\int_{x_0}^{x_n} f(x)dx$$

where it is assumed that $f(x)$ is approximated by the corresponding $p(x)$ on each interval $[x_i, x_{i+1}]$ for $i = 0, 1, \ldots, n-1$ and that all of the intervals have the same length h.

(f) Derive an error formula for the composite quadrature rule in part **e**.