Doctoral Qualifying Exam: Linear Algebra, Probability Distributions and Statistical Inference

Thursday, January 14, 2007

1. Given the 4×2 matrix:

$$\mathbf{A} = \begin{pmatrix} 5 & 3 \\ 3 & 5 \\ 5 & 3 \\ 3 & 5 \end{pmatrix}.$$

   Find bases for the four fundamental subspaces. Find the singular value decomposition (SVD) of this matrix and decompose the matrix into the form $\mathbf{A} = \sum_{i=1}^{2} \alpha_i \mathbf{u}_i \mathbf{v}_i^T$, where $\mathbf{u}_1^T = (1, 1, 1, 1)$, $\mathbf{u}_2^T = (1, -1, 1, -1)$,$\mathbf{v}_1^T = (1, 1)$, and $\mathbf{v}_2^T = (1, -1)$.

2. (a) Based on the relations between the columns of the following matrix, find one eigenvalue and one eigenvector of $\mathbf{A}$:

$$\mathbf{A} = \begin{pmatrix} 0.2 & 0.3 & 0.4 \\ 0.4 & 0.4 & 0.4 \\ 0.4 & 0.3 & 0.2 \end{pmatrix}.$$

   (b) Note that $\mathbf{A}$ is also a Markov matrix, find the other eigenvalues of $\mathbf{A}$.

   (c) If $\mathbf{u}_o = (0 \quad 10 \quad 0)^T$, find the limit of $\mathbf{A}^k \mathbf{u}_o$ as $k \to \infty$.

3. Let $\mathbf{A}$ be an $n \times n$ non-Hermitian matrix. Show that the eigenvalues of $\mathbf{A}^H$ are $\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n$. If $\mathbf{u}_j$ is an eigenvector of $\mathbf{A}$ belonging to λ_j, and if $\mathbf{v}_k$ is an eigenvector of $\mathbf{A}^H$ belonging to $\bar{\lambda}_k$, suppose $\mathbf{A}$ has distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, show that $(\mathbf{u}_j, \mathbf{v}_k) = 0$ if $j \neq k$.

4. (a) If f is the p.d.f. of a random variable (r.v.) T, where $T > 0$ w.p. 1; then prove that

$$g(x, y) = \begin{cases} \frac{f(x+y)}{x+y}, & \text{if } x > 0, \ y > 0 \\ 0, & \text{elsewhere} \end{cases}$$

   is the joint p.d.f. of a random vector (X, Y) on the plane. If the r-th moment of T is finite for some $r > 0$, find EX^r in terms of the moments of T.

   (b) In part (a) above, if $r \geq 2$, find the means and variances of X and Y, and compute the correlation between X and Y in terms of the mean (μ) and variance (σ^2) of T.

5. (a) Using the approximation,

$$P(u < U \leq u + \Delta u, \ \ v < V \leq v + \Delta v) \approx h(u, v)\Delta u \Delta v, \qquad \text{for small } \Delta u, \ \Delta v$$

for any random vector (U, V) with a joint p.d.f. $h(u, v)$; provide the heuristic arguments and the corrsponding derivation of the joint density of the *order statistics* $(X_{(i)}, X_{(j)})$ with $1 \le i < j \le n$ derived from a random sample$(X_1, X_2, \cdots, X_n)$ of size n drawn from a p.d.f. $f(x)$.

(b) A set of r.v.s $(Y_1, Y_2, \cdots, Y_n)$ are said to be exchangable if their joint distribution is invariant under any permutation of the components $Y_i^{'}$s. Show that if $(Y_1, Y_2, \cdots, Y_n)$ are exchangable and positive, then

$$E\left(\frac{Y_{i_1} + Y_{i_2} + \cdots + Y_{i_k}}{Y_1 + Y_2 + \cdots + Y_n}\right) = \frac{k}{n}, \quad k = 1, 2, \cdots, n$$

for any subset $\{i_1, i_2, \cdots, i_k\} \subset \{1, 2, \cdots, n\}$; $1 \le k \le n$.

6. We have a random sample of size $n \ge 2$ from a Poisson distribution of a r.v. X with unknown mean $\lambda > 0$. We need to estimate the parameter $\theta := \{P(X = 0)\}^2$.

   a) Suggest an unbiased estimator of θ and prove its unbiasedness.

   b) Use your estimator in a) above to construct a uniformly minimum variance unbiased (UMVU) estimator of θ. Is it unique ? If so, why ?

   c) What is the large sample distribution of the maximum likelihood estimator of θ? Construct a corresponding large sample confidence interval for θ.