Doctoral Qualifying Exam: Linear Algebra and Numerical Methods

Thursday, January 14, 2007

1. Given the 4×2 matrix:

$$\mathbf{A} = \begin{pmatrix} 5 & 3 \\ 3 & 5 \\ 5 & 3 \\ 3 & 5 \end{pmatrix}.$$

   Find bases for the four fundamental subspaces. Find the singular value decomposition (SVD) of this matrix and decompose the matrix into the form $\mathbf{A} = \sum_{i=1}^{2} \alpha_i \mathbf{u}_i \mathbf{v}_i^T$, where $\mathbf{u}_1^T = (1,1,1,1)$, $\mathbf{u}_2^T = (1,-1,1,-1)$,$\mathbf{v}_1^T = (1,1)$, and $\mathbf{v}_2^T = (1,-1)$.

2. (a) Based on the relations between the columns of the following matrix, find one eigenvalue and one eigenvector of $\mathbf{A}$:

$$\mathbf{A} = \begin{pmatrix} 0.2 & 0.3 & 0.4 \\ 0.4 & 0.4 & 0.4 \\ 0.4 & 0.3 & 0.2 \end{pmatrix}.$$

   (b) Note that $\mathbf{A}$ is also a Markov matrix, find the other eigenvalues of $\mathbf{A}$.

   (c) If $\mathbf{u}_o = (0 \quad 10 \quad 0)^T$, find the limit of $\mathbf{A}^k \mathbf{u}_o$ as $k \to \infty$.

3. Let $\mathbf{A}$ be an $n \times n$ non-Hermitian matrix. Show that the eigenvalues of $\mathbf{A}^H$ are $\bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_n$. If $\mathbf{u}_j$ is an eigenvector of $\mathbf{A}$ belonging to λ_j, and if $\mathbf{v}_k$ is an eigenvector of $\mathbf{A}^H$ belonging to $\bar{\lambda}_k$, suppose $\mathbf{A}$ has distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, show that $(\mathbf{u}_j, \mathbf{v}_k) = 0$ if $j \neq k$.

4. (a) For the iteration scheme $x_{n+1} = 1 + x_n - \frac{x_n^2}{2}$, find the fixed point(s) and determine its (their) stability.

   (b) The Chebyshev polynomials T_n are defined by $T_n(x) = \cos(n \cos^{-1}(x))$. Show that T_n can also be written as

$$T_n(x) = \frac{1}{2}\left[\left(x + \sqrt{x^2 - 1}\right)^n + \left(x - \sqrt{x^2 - 1}\right)^n\right].$$

5. (a) Determine the degree of precision of the approximation

$$\int_0^1 f(x)dx \approx \frac{1}{4} f(0) + \frac{3}{4} f\left(\frac{2}{3}\right).$$

(b) Determine the value of the constants a, b and c in the following formula

$$\int_{-1}^{1} f(x)dx \approx af(-1) + bf(0) + cf(1)$$

so that it is exact for all polynomials of as large a degree as possible. What is the degree of precision of this approximation?

(c) Find a polynomial $p(x)$ of degree ≤ 2 that satisfies

$$p(x_0) = y_0, \quad p(x_1) = y_1, \quad p'(x_0) = y_0'$$

Give a formula in the form

$$p(x) = y_0 l_0(x) + y_1 l_1(x) + y_0' l_2(x)$$

6. (a) The following Runge-Kutta method

$$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right]$$

is used to obtain a numerical solution to $y' = f(t, y)$. Show that this numerical scheme is second order accurate.

(b) What is the order of the following multistep method? Is it convergent (assuming that both y and f are sufficiently smooth)?

$$y_{n+3} - y_n = h\left[\frac{3}{8}f(t_{n+3}, y_{n+3}) + \frac{9}{8}f(t_{n+2}, y_{n+2}) + \frac{9}{8}f(t_{n+1}, y_{n+1}) + \frac{3}{8}f(t_n, y_n)\right].$$