Doctoral Qualifying Exam, Applied Mathematics.

August 30, 1999.

1. In a biological application, the population P of organisms at time t obeys the logistic equation

$$\frac{dP}{dt} = aP\left(1 - \frac{P}{E}\right),$$

where t is time and a and E are constants.

(a) Determine the equilibrium population levels.

(b) Determine their stability.

(c) With the information found in (a) and (b), discuss the qualitative behavior of the population. Explain your assertions by considering the levels of P at which P is increaing and decreasing, respectively.

(d) Solve the diferential equation exactly and compare the results with your answers to parts (a), (b), and (c).

2. Consider the boundary value problem

$$\begin{gathered} u'' = f(x), \quad 0 < x < 1 \\ \epsilon u(0) + u'(0) = 0, \quad u'(1) = 0 \end{gathered}$$

where primes denote differentiation with respect to x.

(a) Find the Green's function for this problem.

(b) Use the Green's function to write down the solution of the boundary value problem.

(c) What happens to the solution as $\epsilon \to 0$? Does the boundary value problem still have a solution?

3. Solve the initial boundary value problem

$$\begin{gathered} u_t - u_{xx} = 0, \qquad x \in (0, \infty), \quad t > 0 \\ u(0,t) = 1 + b\sin\omega t, \quad t > 0; \qquad u(x,0) = 0, \quad x > 0 \end{gathered}$$

where $b > 0$. What is the behavior of the solution as $t \to \infty$?

4. Let $D = \{(x, y, z) \mid 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$ be a 'cubical resonator' and let ∂D denote its sides.

(a) Find the Green's function satisfying

$$G_{tt} = \nabla^2 G + \delta(\boldsymbol{x} - \boldsymbol{x}_0)\delta(t - t_0) \qquad \boldsymbol{x} \in D$$
$$G = G_t = 0, \quad t < t_0; \qquad G = 0, \quad \boldsymbol{x} \in \partial D$$

where $\boldsymbol{x}$ and $\boldsymbol{x}_0$ are in D.
(b) Solve the initial-boundary value problem

$$u_{tt} = \nabla^2 u \qquad \boldsymbol{x} \in D, t > 0$$
$$u = u_t = 0, \quad t = 0; \qquad u(x, y, 0, t) = \sin(\nu t), \quad (x, y) \in \Omega; \qquad u = 0, \quad \boldsymbol{x} \in \partial D - \Omega$$

where Ω is a simply connected region on the face $z = 0$ of the cube. What happens when $\nu \to \pi\sqrt{l^2 + m^2 + n^2}$ where $l, m,$ and n are integers?

5. Let R be the interior of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and let ∂R be its surface. Solve the boundary value problem

$$\nabla^2 u = 0 \qquad (x, y, z) \in R$$
$$u = 1 \qquad (x, y, z) \in \partial R$$

and prove that the solution is unique.

6. Consider the eigenvalue problem

$$\frac{1}{x^2}(x^2 u')' + \lambda u = 0, \qquad 0 < x < 1$$
$$|u(0)| < \infty, \qquad u(1) = 0$$

where primes denote differentiation with respect to x.

(a) Show that the eigenvalues λ are real and positive.

(b) Use the Rayleigh-Ritz theorem and an appropriate quadratic polynomial trial function to find an approximation of the first zero of $j_0(x)$, where j_0 is the spherical Bessel function of the first kind and order zero which is bounded at the origin. Justify your choice of trial function. You can quote the following facts about Bessel functions: the equation $(x^2u')' + x^2u = 0$ has independent solutions j_0 and y_0. The solution j_0 is bounded at $x = 0$ while y_0 is not, with $j_0(0) = 1$ and $j_0'(0) = 0$.