Doctoral qualifying exam:
Applied Mathematics.

September 2, 1997.

You have three hours for this exam. Show all working in the books provided.

1. Construct the Green's function $G(x,\xi)$ for the initial value problem

$$u'' + 3u' + 2u = f(x) \quad 0 < x < \infty$$
$$u(0) = 0, \quad u'(0) = 1.$$

Explain why $G(x,\xi) \equiv 0$ for $0 < x < \xi$ and write down the solution for u in terms of f.

2. Show that the boundary value problem

$$u'' + u' = f(x) \quad 0 < x < 1$$
$$u(0) + u'(0) = 0, \quad u(1) + u'(1) = 0$$

has a nonzero homogeneous solution and derive the consistency condition necessary for a solution to exist. Construct the modified Green's function and give the general form of the solution.

3. Write down the Green's function and the solution to the initial boundary value problem

$$u_t - u_{xx} = p(x,t) \quad x \in (0,\infty) \;\; t > 0$$
$$u_x(0,t) = g(t) \;\; t > 0, \quad u(x,0) = f(x) \;\; x > 0.$$

Show that the contribution to the solution from the boundary data alone can be written

$$u = \frac{-1}{\sqrt{\pi}} \int_0^t g(t-\tau) \frac{e^{-\frac{x^2}{4\tau}}}{\tau^{\frac{1}{2}}}\, d\tau.$$

4. Let $D = \{(x,y,z) \mid 0 \le x \le 1,\; 0 \le y \le 1,\; 0 \le z \le 1\}$ be a 'cubical resonator' and let ∂D denote its sides.
(a) Find the Green's function satisfying

$$G_{tt} = \nabla^2 G + \delta(\mathbf{x} - \mathbf{x_0})\delta(t - t_0), \quad \mathbf{x} \in D$$
$$G = G_t = 0,\; t < t_0$$
$$G = 0, \quad \mathbf{x} \in \partial D$$

where $\mathbf{x}$ and $\mathbf{x_0}$ are in D.

(b) Solve the initial-boundary value problem

$$
\begin{gathered}
u_{tt} = \nabla^2 u \quad \mathbf{x} \in D,\ t > 0 \\
u = u_t = 0, \quad t = 0 \\
u(x, y, 0, t) = f(x, y) \sin(\nu t), \quad (x, y) \in \Omega \\
u = 0, \quad \mathbf{x} \in \partial D - \Omega
\end{gathered}
$$

where Ω is a simply connected region on the face $z = 0$ of the cube.

5. Let D be the region in the upper half-plane that is exterior to the unit semi-circle and let C denote the boundary of this infinite region.
(a) Find the Green's function $G(x, y|x', y')$ satisfying

$$
\begin{gathered}
\nabla^2 G = \delta(x - x')\delta(y - y'), \quad (x, y) \in D \\
\frac{\partial G}{\partial n} = 0, \quad (x, y) \in C \\
G \sim a \ln r, \quad r \to \infty.
\end{gathered}
$$

(b) Solve the potential problem

$$
\begin{gathered}
\nabla^2 u = 0, \quad (x, y) \in D \\
\frac{\partial u}{\partial n} = 0, \quad (x, y) \in C \\
u \sim x, \quad r \to \infty
\end{gathered}
$$

by first transforming the dependent variable to v where $u = x + v$.

6. Inviscid fluid with density ρ_2 occupies the region $y > 0$ and translates with uniform velocity $\mathbf{u} = U\mathbf{i}$ over heavier fluid with density ρ_1 which lies at rest in $y < 0$.

Examine the linear stability of this flow as follows. Let the disturbed free surface be at $y = \eta(x, t)$ and the fluid velocity be

$$
\mathbf{u} = U\mathbf{i} + \nabla\phi_2 \text{ for } y > \eta, \qquad \mathbf{u} = \nabla\phi_1 \text{ for } y < \eta.
$$

(a) Explain why it is reasonable to model the flow by potential flow, and why the potentials satisfy Laplace's equation,

$$
\nabla^2\phi_2 = 0 \quad y > 0, \qquad \nabla^2\phi_1 = 0 \quad y < 0.
$$

(b) Find the linearized form of the kinematic boundary condition, which is $\frac{D}{Dt}(y - \eta) = 0$ on $y = \eta$.
(c) The Bernoulli equation $\partial_t\phi + (p/\rho) + (\mathbf{u}.\mathbf{u}/2) + gy = G(t)$ holds throughout each fluid, and the jump in pressure across $y = \eta$ is $p_2 - p_1 = T\partial_x^2\eta$, where T is the surface tension and g is the acceleration due to gravity. Use these to find the linearized boundary condition

$$
\rho_1\Big(\frac{\partial\phi_1}{\partial t} + g\eta\Big) - \rho_2\Big(\frac{\partial\phi_2}{\partial t} + U\frac{\partial\phi_2}{\partial x} + g\eta\Big) = T\frac{\partial^2\eta}{\partial x^2}.
$$

(d) Look for disturbances proportional to $\exp i(kx - \omega t)$ which decay as $|y| \to \infty$. Show that these satisfy the dispersion relation

$$(\rho_1 + \rho_2)\omega^2 - (2\rho_2 U k)\omega + \left(\rho_2 U^2 k^2 - k(Tk^2 + (\rho_1 - \rho_2)g)\right) = 0$$

and hence that the interface is unstable when

$$U^2 > \frac{\rho_1 + \rho_2}{\rho_1 \rho_2}\left(Tk + (\rho_1 - \rho_2)\frac{g}{k}\right).$$