Doctoral Qualifying Examination
in
Applied Mathematics

Part C: Applied Mathematics

September 14, 1994

Problem 1.

(a) Find the exact eigenvalues λ_n and eigenfunctions $y_n(x)$ of the eigenvalue problem

$$\begin{aligned} y'' + \lambda y &= 0 \\ y(0) &= 0 \\ y(1) &= 0 \end{aligned}$$

(b) Show how to use the Ritz' method to obtain an upper bound for each of the first two eigenvalues of the same problem using admissible *polynomial* test functions of degree 3 or less.

Problem 2. Consider the eigenvalue problem

$$\frac{d^2u}{dx^2} + \lambda u = 0, \qquad 0 < x < 1 \tag{2a}$$

$$\frac{du(0)}{dx} + \lambda u(0) = 0, \qquad u(1) = 0 \tag{2b}$$

where the eigenvalue parameter appears explicitly in the boundary condition (2b).

(a) Find the eigenvalues and corresponding eigenfunctions (graphical solutions are legitimate).

(b) Show that

$$\int_0^1 u_m(x) u_n(x)\, dx \neq \delta_{mn}$$

where u_m and u_n are two distinct eigenfunctions.

(c) Find the appropriate scalar product so that the eigenfunctions form an orthonormal set.

Problem 3. Using the method of characteristics (or other appropriate method), solve the initial value problem

$$\epsilon\left(\frac{\partial^2}{\partial t^2}u - \frac{\partial^2}{\partial x^2}u\right) = \frac{\partial}{\partial x}u - \frac{\partial}{\partial t}u, \quad |x| < \infty, \quad t > 0$$

$$u(x,0) = f(x)$$

$$\frac{\partial}{\partial t}u(x,0) = g(x)$$

where $\epsilon > 0$. What happens to the solution as $\epsilon \to 0$?

Problem 4. Consider the Helmholtz equation

$$\nabla^2 u + k^2 u = 0$$

in three dimensions. You may find it helpful to recall that the Laplacian in spherical coordinates is given by

$$\nabla^2 f \;=\; \frac{1}{r^2}\left(r^2 f_r\right)_r \;+\; \frac{1}{r^2 \sin\theta}\left(\sin\theta f_\theta\right)_\theta \;+\; \frac{1}{r^2 \sin^2\theta} f_{\phi\phi}.$$

(a) Find the free space Green's function (sometimes called the fundamental solution).

(b) Use the method of images to solve

$$\nabla^2 v + k^2 v = 0, \qquad |x| < \infty, \quad |y| < \infty, \quad z > 0$$

$$v(x,y,0) = f(x,y)$$

where f is zero outside of some compact region D.

(c) Find the behavior of v as $r = \sqrt{x^2+y^2+z^2} \to \infty$. This is an aperture problem which arise in acoustics and electromagnetics.

Problem 5. A uniform jet of gas with speed U is directed vertically upwards, and a small spherical particle with constant density ρ_s is placed in the jet.

The particle size decreases in time but its shape is always spherical with a radius $a = a(t)$ which is known. Note that the decrease in size implies that the mass of the particle also changes in time.

If we assume that the motion of the particle relative to the gas obeys Stokes' law, then the fluid exerts a drag force on the particle which is $6\pi\rho_g\nu_g a(t)$ multiplied by the *velocity* of the particle *relative to the gas*, where ρ_g and ν_g are the constant gas density and kinematic viscosity.

Gravity acts downwards, with acceleration g.

(a) Set up the balance of forces on the particle and apply Newton's second law to find the second order ODE for the vertical displacement $x = x(t)$ of the particle from the point at which it starts.

(b) If the particle radius satisfies the equation

$$\frac{da}{dt} = -\frac{\chi}{a}$$

where $\chi > 0$ is constant and $a(0) = a_0$, find $a = a(t)$.

(c) Use the lifetime of a particle to introduce a non-dimensional time, and find a suitable non-dimensionalization for distance that simplifies the ODE found in (a).

(d) Describe the motion of the particle just before its radius goes to zero.