Ph.D Qualifying Exam in Applied Mathematics

August 27, 2002

Problem 1.

An inextensible heavy string or cable of mass density ρ is supported under gravity (g) between points $(-a, 0)$ and $(a, 0)$ and has length $2l$ where $l > a$. If the tension in the cable is σ, resolve forces in the horizontal direction to show that $\frac{d\sigma}{d\phi} = \sigma \tan\phi$ and hence that $\sigma = \sigma_0 \sec\phi$, where ϕ is the angle between the tangent to the cable and the horizontal at any point and σ_0 is the tension at $x = 0$. Resolve forces in the vertical direction to show that $\beta \frac{ds}{d\phi} = \sec^2\phi$ where $\beta = \frac{\rho g}{\sigma_0}$ and s is arc length. Hence find the equation of the curve (a 'catenary') for the cable. What happens to the tension in the cable as $l \to a$.

Very briefly, describe how you would derive an equation governing small-amplitude disturbances on the cable, and what form of equation you would expect to find.

Problem 2.

Consider a linear self-adjoint boundary value problem $Lu = f(x)$ on $x \in (0, 1)$ with linear homogeneous boundary conditions at $x = 0, 1$, e.g. $u(0) = u(1) = 0$. Let λ_n and u_n be the eigenvalues and eigenfunctions of $Lu = \lambda u$, and suppose that $\lambda = 0$ is not an eigenvalue of the system.

(a) Show that the solution u has an eigenfunction expansion

$$u(x) = \sum_1^\infty \frac{\alpha_n u_n(x)}{\lambda_n},$$

where $\alpha_n = < u_n, f >$ and $< \, , \, >$ is the inner product.

(b) Use the result of (a), stating any assumptions you need to make, to show that the Green's function has eigenfunction expansion

$$G(x, \xi) = \sum_1^\infty \frac{u_n(x)\, u_n(\xi)}{\lambda_n}.$$

(c) Use the result of (b) to deduce (at least formally) that

$$\delta(x - \xi) = \sum_1^\infty u_n(x)\, u_n(\xi).$$

Hence show that

$$\int_0^1 G(x, x)\, dx = \sum_1^\infty \frac{1}{\lambda_n}.$$

Problem 3.
Consider the boundary value problem

$$u'' = f(x) \quad x \in (0,1)$$

$$u'(0) + \epsilon u(0) = 0 \quad u'(1) = 0$$

where primes denote derivatives with respect to x.

(a) Construct the Green's function for this problem.

(b) Write the solution to the boundary value problem in terms of the Green's function.

(c) What happens as $\epsilon \to 0$? Does the boundary value problem have a solution? Carefully explain your answer.

Problem 4.
Consider the heat flow problem

$$u_t = u_{xx} \qquad 0 < x < 1, \quad t > 0,$$

$$u(x,0) = 0,$$

$$u(0,t) = 0, \quad u(1,t) + u_x(1,t) = 1, \quad t > 0.$$

(a) Find the solution using separation of variables.

For what times t is a truncation of this series solution a good approximation?

(b) We can also solve this problem by taking Laplace transforms in time. Find the Laplace transform $U(s)$ of the problem above, where s is the Laplace transform variable. Instead of attempting an inversion, expand for large s and invert term-by-term. This gives an alternative series representation to that found in part (a). [You may quote the result that for $t > 0$ and λ real, the Laplace transform of $erfc\left(\frac{\lambda}{2\sqrt{t}}\right)$ is equal to $\frac{1}{s}e^{-\lambda\sqrt{s}}$.]

For what times t is a truncation of this series solution a good approximation?

Problem 5.

(a) Let G satisfy the boundary value problem

$$\nabla^2 G = -\delta(\mathbf{x} - \mathbf{x}'), \quad \mathbf{x}, \mathbf{x}' \in \mathbf{\Omega} \tag{1a}$$

$$G = 0, \quad \mathbf{x} \in \partial\mathbf{\Omega} \tag{1b}$$

where Ω is a compact region in three-dimensional space with a smooth boundary $\partial\Omega$. Physically, G represents the temperature produced by a point source at $\mathbf{x} = \mathbf{x}'$ with the boundary of Ω held at a fixed temperature.

Show that $G > 0$ in Ω and $\frac{\partial G}{\partial n} < 0$ on $\partial\Omega$ where $\frac{\partial G}{\partial n}$ denotes the outward going normal derivative.

(b) Consider the function H which satisfies the related problem

$$\nabla^2 H = -\delta(\mathbf{x} - \mathbf{x}'), \quad \mathbf{x}, \mathbf{x}' \in \mathbf{\Omega} \tag{2a}$$

$$\frac{\partial H}{\partial n} + hH = 0, \quad \mathbf{x} \in \partial\mathbf{\Omega} \tag{2b}$$

where $h > 0$ is a positive constant. Physically, H represents the temperature produced by a point source at $\mathbf{x}'$. The boundary is now subjected to cooling, and this is modeled by (2b).

Show that $H > G$ for all $\mathbf{x} \in \mathbf{\Omega}$, where G is the solution of (1a-b).

[Hint: Consider the function $\Phi = H - G$. What follows if you can prove that $H > 0$ on $\partial\Omega$? To prove this argue by contradiction, i.e. assume that $H < 0$ on all or part of the boundary and work out what happens. You may find the identity $\nabla \cdot (H\nabla H) = |\nabla H|^2 + H\nabla^2 H$ useful.]

Problem 6.

(a) Show that the free-space Green's function for the three-dimensional wave equation, that is, the solution of

$$G_{tt} - c^2(G_{xx} + G_{yy} + G_{zz}) = \delta(x - \xi_1)\delta(y - \xi_2)\delta(z - \xi_3)\delta(t - \tau),$$

$$G = G_t = 0 \quad \text{for} \quad t < \tau,$$

is

$$G(x, y, z, t; \xi_1, \xi_2, \xi_3, \tau) = \frac{1}{4\pi c^2 r}\delta(t - \tau - r/c),$$

where $r^2 = (x - \xi_1)^2 + (y - \xi_2)^2(z - \xi_3)^2$. Here, c is a positive constant. [Note that the spherically symmetric Laplace operator can be written as $\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}(\cdot) + (\cdot)\right)$]

Describe this solution physically.

(b) We want to calculate the form of the disturbance u resulting from a unit source moving with fixed velocity V along the positive x-axis. Explain how the problem to be solved now becomes

$$u_{tt} - c^2(u_{xx} + u_{yy} + u_{zz}) = \delta(x - Vt)\delta(y)\delta(z), \tag{U}$$

and use superposition to show that the solution can be written as

$$u = \frac{1}{4\pi c^2}\int_{-\infty}^{\infty} \frac{\delta\left[t-\tau-(1/c)[(x-V\tau)^2+y^2+z^2]^{1/2}\right]}{[(x-V\tau)^2+y^2+z^2]^{1/2}}d\tau.$$

[Hint: To do this, represent the source on the right hand side of equation (U) as

$$\delta(x-Vt)\delta(y)\delta(z) = \delta(y)\delta(z)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(\xi_1-V\tau)\delta(t-\tau)\delta(x-\xi_1)d\xi_1 d\tau,$$

and then use the principle of superposition.]

(c) For the case $V < c$, change variables using

$$\lambda = \tau + (1/c)[(x-V\tau)^2+y^2+z^2]^{1/2},$$

to show that the solution takes the more compact form

$$u(x,y,z,t) = \frac{1}{4\pi c^2}\left[(x-Vt)^2 + \left(1-\frac{V^2}{c^2}\right)(y^2+z^2)\right]^{-1/2}.$$

What does the limit $V \to 0$ recover?

(d) Use the solution in (c) above to find and characterize the shapes of constant u, in a frame of reference moving with the source. What happens as $V \to c-$?