Doctoral Qualifying Exam, Applied Mathematics.

May 19, 1999.

You have three hours for this exam. Show all working in the books provided.

1. A spherical drop of constant density ρ falls under gravity and passes through moist air, so that moisture condenses onto the drop surface from its surroundings and the drop grows. The time rate of change of the drop's mass due to condensation is proportional to its surface area, with constant of proportionality α. Assume that the drop experiences a drag force proportional to the product of the drop velocity and its surface area, with constant of proportionality β.

(i) Write down equations that govern the mass $m(t)$ of the drop, its radius $r(t)$, and its velocity $\dot{x}(t) = v(t)$. Here t denotes time and x is the position of the particle measured from a suitable reference point.

(ii) What are the dimensions of the parameters α and β?

(iii) Show that if condensation begins at time $t = 0$ with initial drop radius r_0, then at subsequent times $r(t) = \frac{\alpha}{\rho}t + r_0$.

(iv) Use the result above to obtain the velocity of the drop, and deduce that at large times the velocity is given by $v(t) \sim -(gt)/(3(\beta/\alpha) + 4)$.

(v) The last result shows that the velocity increases without bound as the particle falls. Can you suggest a new drag force law which would produce a finite terminal velocity? Explain your suggestion.

2. For the eigenvalue problem

$$Lu \equiv \frac{1}{x^2}\frac{d}{dx}\left(-x^2\frac{du}{dx}\right) + \frac{n(n+1)}{x^2}u = \lambda u \qquad x \in (0,1)$$

$$B_1 u = \lim_{x\to 0} x\frac{du}{dx} = 0, \qquad B_2 u = u(1) = 0,$$

show that the operator $\mathcal{L} = (L, \mathcal{D}_{\mathcal{B}})$ is self-adjoint and positive definite with respect to the inner product $< u, v >= \int_0^1 uvx^2\,dx$ with weight x^2. What does this imply about the eigenvalues of $\mathcal{L}$? Form the Rayleigh quotient $\rho(u) =< u, Lu > / < u, u >$ and state the result that enables you to use this to approximate the eigenvalues of $\mathcal{L}$.

Given that $u = j_n(\sqrt{\lambda}x)$ are the eigenfunctions where λ is an eigenvalue, and that when $n = 0$, $j_0(0) = 1$ with $j_0'(0) = 0$, suggest simple polynomial trial functions that enable you to find the first two zeros of $j_0(z)$. Explain how you would use these trial functions, but you do not have to perform any detailed calculations with them.

3. Find the Green's function $G(x, \xi; \mu)$ defined by

$$-\frac{d^2G}{dx^2} - \mu G = \delta(x-\xi) \qquad x, \xi \in (0, l)$$
$$B_1 G = G(0, \xi; \mu) = 0, \qquad B_2 G = G(l, \xi; \mu) = 0,$$

where μ is not an eigenvalue. Show that this can be written

$$G(x, \xi; \mu) = \frac{\sin\sqrt{\mu}x_< \sin\sqrt{\mu}(l - x_>)}{\sqrt{\mu}\sin\sqrt{\mu}l} \qquad 0 \leq x, \xi \leq l$$
$$\text{where } x_< = \min(x, \xi) \quad \text{and} \quad x_> = \max(x, \xi).$$

Consider $G(x, \xi, \mu)$ as a function of the complex parameter μ:
(i) Show that $G(x, \xi, \mu)$ has simple poles at a sequence of points $\mu = \mu_n$, $n = 1, 2, ...$, and find the points μ_n.
(ii) Find the residue of $G(x, \xi; \mu)$ at $\mu = \mu_n$.
State the relation between your answers to (i) and (ii) and the eigenvalues, $\lambda = \lambda_n$, and eigenfunctions, $u = u_n$, of the problem

$$-\frac{d^2u}{dx^2} = \lambda u \qquad x \in (0, l)$$
$$u(0) = 0, \qquad u(l) = 0.$$

4. The potential $u_q(\boldsymbol{x})$ due to a surface distribution of sources of strength $q(\boldsymbol{x})$ and the potential $u_p(\boldsymbol{x})$ due to a surface distribution of dipoles of strength $p(\boldsymbol{x})\boldsymbol{n}(\boldsymbol{x})$ on a surface S are given by

$$u_q(\boldsymbol{x}) = \frac{-1}{4\pi}\int\int_{\boldsymbol{\xi}\in S} \frac{q(\boldsymbol{\xi})}{|\boldsymbol{x}-\boldsymbol{\xi}|} dS_{\boldsymbol{\xi}},$$
$$u_p(\boldsymbol{x}) = \frac{-1}{4\pi}\int\int_{\boldsymbol{\xi}\in S} p(\boldsymbol{\xi})\boldsymbol{n}(\boldsymbol{\xi}).\nabla_{\boldsymbol{\xi}}\left(\frac{1}{|\boldsymbol{x}-\boldsymbol{\xi}|}\right) dS_{\boldsymbol{\xi}},$$

respectively, where $\boldsymbol{n}(\boldsymbol{x})$ is the (outward) unit normal on S.

If S is the unit sphere, find the integrals for u_q and u_p using spherical polar coordinates $\boldsymbol{x} = (r, \theta, \phi)$ and $\boldsymbol{\xi} = (\rho, \theta', \phi')$. Use these to show that the potential due to a surface distribution consisting of both dipoles $2f(\boldsymbol{x})\boldsymbol{n}(\boldsymbol{x})$ and sources $f(\boldsymbol{x})$ is

$$u = \frac{1-r^2}{4\pi}\int_0^{2\pi}\int_0^{\pi} \frac{f(\theta', \phi')\sin\theta'}{(1+r^2-2r\cos\gamma)^{\frac{3}{2}}} d\theta' d\phi'$$

where γ is the angle between $\boldsymbol{x}$ and $\boldsymbol{\xi}$.

Show that this is the same as the solution to the interior Dirichlet problem for a sphere,

$$\begin{aligned} \nabla^2 u = 0 \qquad & |\boldsymbol{x}| < 1 \\ u(\boldsymbol{x}) = f(\theta, \phi) \quad & \text{on } |\boldsymbol{x}| = 1. \end{aligned}$$

5. For the boundary value problem

$$\begin{aligned} u_{tt} - u_{xx} = p(x,t) \qquad & x \in (0,1), \quad t > 0 \\ u(x,0) = f(x), \qquad & u_t(x,0) = g(x), \\ u(0,t) = l(t), \qquad & u_x(1,t) = m(t), \end{aligned}$$

find the Green's function $G(x, \xi, t, \tau)$ by the method of images, and sketch G with its image system in the x,t-plane. Find the eigenfunction expansion of the Green's function. Find the representation of the solution for $u(x,t)$ in terms of the Green's function.

6. A viscous fluid is initially at rest between two infinite planes at $y = 0, h$. At time $t = 0$ the plane $y = h$ is set in motion impulsively, so that it has constant velocity U in the x-direction for $t > 0$. Show that after the impulse, the velocity in the fluid is given by

$$\frac{Uy}{h} + \sum_{n=1}^{\infty} \frac{2U}{n\pi}(-1)^n \sin\frac{n\pi y}{h} \exp\left(-\frac{n^2\pi^2}{h^2}\nu t\right),$$

where ν is the kinematic viscosity of the fluid.

Hence show that the stress per unit area on the plane $y = h$ decreases monotonically in t from the initial impulse to an ultimate value of $U\mu/h$.