Doctoral Qualifying Exam: Applied Mathematics.

Thursday, May 21, 1998.

You have three hours for this exam. Show all working in the books provided.

1. Consider the boundary value problem

$$u'' - 2u' + u = f(x) \qquad x \in (0,1)$$

$$u'(0) + \alpha u(0) = c_1 \qquad u(1) = c_2$$

where the parameter α is real. For what values of α does a Green's function $G(x,\xi)$ exist? Construct the Green's function when it exists and give the solution of the boundary value problem $u(x)$ in terms of it. What condition must be satisfied by the data f, c_1, c_2 to ensure that the solution for u remains bounded as $\alpha \to 0$?

2. Find the eigenvalues and eigenfunctions associated with the boundary value problem

$$u'' = f(x) \qquad x \in (0,1)$$

$$u(0) = \alpha \qquad u(1) = \beta.$$

Express the solution of the boundary value problem as a series in terms of the eigenfunctions. Comment on the convergence of your series to the solution of the boundary value problem near the end-points $x = 0$ and $x = 1$ when α and β are non-zero. Give the solution for u if $\alpha = \beta = 0$ and $f(x) = \cos \pi x \sin 3\pi x$.

3. The temperature u of a semi-infinite rod satisfies

$$u_t - u_{xx} = 0 \quad x \in (0,\infty), \ t > 0$$

$$u(x,0) = 0 \quad u(0,t) = g(t).$$

Show that the solution u can be written as

$$u(x,t) = \frac{x}{2\sqrt{\pi}} \int_0^t \frac{e^{-\frac{x^2}{4\tau}}}{\tau^{3/2}} \, g(t-\tau)\, d\tau.$$

Show also that as $x \to \infty$ for fixed $t > 0$

$$u(x,t) \sim \frac{2\sqrt{t}}{\sqrt{\pi}x} e^{-\frac{x^2}{4t}} \left(g(0) - \frac{2t}{x^2}(g(0) - 2tg'(0)) + O(x^{-4}) \right).$$

Hint: you may find some of the following results useful to find the solution u:

$$\frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-\frac{(x-\xi)^2}{4a}}}{\sqrt{4a}}\, d\xi = 1 - \frac{1}{2}\mathrm{erfc}\,(\frac{x}{2\sqrt{a}}) \qquad \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{e^{-\frac{(x+\xi)^2}{4a}}}{\sqrt{4a}}\, d\xi = \frac{1}{2}\mathrm{erfc}\,(\frac{x}{2\sqrt{a}})$$

where $\mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-s^2}\, ds$. The functions below are a Laplace transform pair.

$$f(x,t) = \frac{e^{-\frac{x^2}{4t}}}{2\sqrt{\pi t}} \qquad\qquad F(x,s) = \frac{e^{-x\sqrt{s}}}{2\sqrt{s}}$$

4. The Neumann problem for Laplace's equation in a half-space is

$$\nabla^2 u = 0 \quad x \in (-\infty,\infty) \quad y \in (-\infty,\infty) \quad z > 0$$

$$\frac{\partial u}{\partial z}(x,y,0) = \begin{cases} f(x,y) & x^2+y^2<1 \\ 0 & x^2+y^2 \geq 1. \end{cases}$$

Find the Green's function for this problem and give the solution for u in terms of it.
Show that far from the origin, as $|\boldsymbol{x}| \to \infty$, the solution for u has the expansion

$$u(\boldsymbol{x}) \sim -\frac{1}{2\pi|\boldsymbol{x}|} \int\!\!\int_\Omega f(\xi,\eta)\, d\xi d\eta - \frac{1}{2\pi|\boldsymbol{x}|^3} \int\!\!\int_\Omega f(\xi,\eta)\, \boldsymbol{x}.\boldsymbol{\xi}_\circ\, d\xi d\eta + O(|\boldsymbol{x}|^{-3})$$

where $\boldsymbol{x} = (x,y,z)$, $\boldsymbol{\xi}_\circ = (\xi,\eta,0)$, and Ω is the interior of the unit circle. Explain why the second term in the expansion is zero when f is constant.

5. Consider the boundary value problem

$$\nabla^2 u + k^2 u = f(x,y) \quad x \in (-\infty,\infty) \quad y \in (0,1)$$

$$\frac{\partial u}{\partial y}(x,0) = \frac{\partial u}{\partial y}(x,1) = 0$$

where $k > 0$ is given. The source term $f(x,y)$ has compact support, that is, $f(x,y)$ is identically zero outside a compact region Ω which is contained completely within D. The solution u also satisfies the boundary condition that it represent outgoing waves as $|x| \to \infty$. Construct the Green's function for this problem and use it to express the solution of the boundary value problem u .
Show that for x positive and sufficiently large

$$u \sim \sum_{n=0}^{M} T_n \cos n\pi y\, e^{ik_n x}$$

where $k_n = \sqrt{k^2 - n^2\pi^2} > 0$ for $n \leq M$, and express the coefficient T_n in terms of f. Explain why the sum terminates when $n = M$, and what this corresponds to physically.

6. In its dimensional form, an initial value problem for Burger's equation is

$$\frac{\partial \hat\rho}{\partial \hat t} + U\frac{\hat\rho}{\hat\rho_1}\frac{\partial \hat\rho}{\partial \hat x} = \nu \frac{\partial^2 \hat\rho}{\partial \hat x^2},$$

$$\hat{\rho}(\hat{x},0) = \begin{cases} \hat{\rho}_1 & x < 0, \\ \hat{\rho}_2 & x > 0, \end{cases}$$

where $\hat{\rho}$ is a density (mass/volume), and $\hat{\rho}_1$, $\hat{\rho}_2$, ν, and U are known or given constants with U being a velocity.
(a) What are the dimensions of ν?
(b) Non-dimensionalize the problem, to write it in the form

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial \rho}{\partial x} = \epsilon \frac{\partial^2 \rho}{\partial x^2},$$

$$\rho(x,0) = \begin{cases} 1 & x < 0, \\ A & x > 0, \end{cases}$$

where the dimesionless parameters are $\epsilon = \nu/\hat{x}^* U$ and $A = \hat{\rho}_2/\hat{\rho}_1$, with $\hat{x}^*$ an arbitrary length scale.
(c) Write down the solution of the reduced problem (i.e., with $\epsilon = 0$) using the method of characteristics. Consider all values of A.
(d) Under what conditions does a shock form? What is the shock speed?