Applied Math Qualifying Exam, Spring 1997

1. Solve the following boundary value problem by constructing an appropriate Green's function.

$$\frac{d}{dx}\Big((1+x)\frac{du}{dx}\Big) = f(x), \qquad 0 < x < 1,$$

$$u(0) = 0, \quad u'(0) = u'(1).$$

2. Consider the boundary value problem

$$u'' + u' = f(x), \qquad x \in (-\infty, 0) \cup (0, \infty),$$

$$u \to 0 \text{ as } x \to \infty, \quad u \text{ grows at most algebraically as } x \to -\infty,$$

$$\text{with jump conditions } [u] = 0, \;\; [u'] + u(0) = \beta \text{ at } x = 0,$$

where $[u] \equiv u(0^+) - u(0^-)$.

- Show that this system has a nonzero homogeneous solution.
- Find the adjoint problem and its homogeneous solution using the inner product $< v, u >= \int_{-\infty}^{0^-} vu\ dx + \int_{0^+}^{\infty} vu\ dx$, then find the necessary condition for a solution of the above BVP to exist.
- Write down the general form of the solution to the above BVP when

$$f(x) = \begin{cases} \nu e^{-x}, & x > 0 \\ 1, & x < 0. \end{cases}$$

3. Consider the initial value problem:

$$u_{tt} - u_{xx} = f(x)g(t), \qquad |x| < \infty, \; t > 0,$$

$$u = u_t = 0, \quad t = 0, \quad |x| < \infty,$$

$$u \text{ bounded as } |x| \to \infty.$$

Here $f(x)$ is identically zero outside the bounded interval $x \in \Omega$, and $g(t) = H(t)\sin(\omega t)$ where $H(t)$ is the Heaviside function. Write down the free-space Green's function, or fundamental solution.

Now let x_1 be a fixed point outside of Ω.

- Show that $u(x_1, t) \equiv 0$, for $t \leq T_1$, where T_1 is the arrival time.
- Show that $u(x_1, t) = v(x_1)\sin(\omega t)$ for $t \geq T_2$. That is, u is periodic after a finite amount of time.
- Give geometrical interpretations of T_1 and T_2.

4. Let D be the channel region, $D = \{(x, y) |\, |x| < \infty, 0 < y < 1\}$. Find the Green's function, G, satisfying

$$yG_{xx} + yG_{yy} + G_y = \delta(x-\xi)\delta(y-\eta), \quad \text{in } D,$$

$$G(x, 1) = 0, \quad |x| < \infty,$$

$$G \text{ bounded as } y \to 0; \qquad G \text{ bounded as } |x| \to \infty.$$

Hint: Express the solution in the form of an infinite series

$$G = \sum_{n=0}^{\infty} g_n(x;\xi) f_n(y;\eta),$$

and recall that Bessel's equation of order zero is

$$(yu')' + yu = 0.$$

5. The BVP

$$(\partial_x^2 + \partial_y^2)\phi = 0, \quad x \in (-\infty, \infty), \quad y \in (-h, 0),$$

$$\text{on } y = 0: \quad -\frac{\partial \eta}{\partial t} + \frac{\partial \phi}{\partial y} = 0, \quad \frac{\partial \phi}{\partial t} - \frac{T_0}{\rho_0}\frac{\partial^2 \eta}{\partial x^2} + g\eta = 0,$$

$$\text{on } y = -h: \quad \frac{\partial \phi}{\partial y} = 0,$$

$$\phi \text{ is bounded as } |x| \to \infty,$$

describes linearized surface waves on water of finite depth h, with surface tension T_0, density ρ_0, and gravitational constant g. The free surface has elevation $y = \eta(x, t)$, and $\phi(x, y, t)$ is the velocity potential so that the fluid velocity is $\mathbf{v} = \nabla\phi$.

Look for harmonic disturbances with (ϕ, η) proportional to $\exp i(kx - wt)$, and show that this gives the dispersion relation

$$\omega^2 = gk \tanh(kh)\Big(1 + \frac{T_0 k^2}{g\rho_0}\Big).$$

Explain why the approximate version of this relation for waves on deep water describes both 'gravity driven' and 'surface tension driven (or capillary)' waves; what distinguishes these two types in terms of (i) wavelength and (ii) their ratio of group speed to phase speed?

Briefly explain why surface water waves are modelled by the BVP above, i.e. explain why the velocity potential is such that $\nabla^2\phi = 0$, and why the boundary conditions take the given form.