Applied Mathematics Qualifying Exam May 15, 1995

1. (a) Find the Green's function $G(x;\xi)$ for the following one-dimensional problem:

$$\frac{d^2G}{dx^2} = \delta(x-\xi), \quad 0<\xi<1, \quad 0<x<1,$$

$$\frac{dG}{dx}(0;\xi) = 0, \quad \epsilon G(1;\xi) + \frac{dG}{dx}(1;\xi) = 0, \quad \epsilon > 0.$$

(b) Use the Green's function found in part (a) to solve the following boundary value problem:

$$\frac{d^2u}{dx^2} = f(x), \quad 0<x<1,$$

$$\frac{du}{dx}(0) = 0, \quad \epsilon u(1) + \frac{du}{dx}(1) = 0, \quad \epsilon > 0, \tag{1}$$

and show that as $\epsilon \to 0$ the solution remains bounded only if $\int_0^1 f(x)dx = 0$.

(c) Invoke the Fredholm alternative to show that for a solution of problem (1) with $\epsilon = 0$ to exist, the condition $\int_0^1 f(x)dx = 0$ must be satisfied. What is the physical interpretation of this condition if $u(x)$ represents temperature in a heat flow problem.

2. Consider the eigenvalue problem

$$y'' + \lambda y = 0,$$

with boundary conditions

$$y(0) - y(1) = 0, \quad y'(0) - y'(1) = 0.$$

(a) Show that the problem is self adjoint (note that it is not of Sturm-Liouville type).

(b) Find **all** eigenvalues and corresponding eigenfunctions.

3. (a) Find the Green's function $G(x,y;\xi,\eta)$ given by

$$G_{xx} + G_{yy} = \delta(x-\xi)\delta(y-\eta) \quad \text{in D},$$

$$G = 0 \quad \text{on } \partial\text{D},$$

where the domain D is the first quadrant of the $x-y$ plane, $x \geq 0,\ y \geq 0$.
Do this by (i) using the method of images, and, (ii) applying the conformal transformation $w = z^2$ to map D to the upper half w-plane, and then using the method of images.

(b) Write down the solution to the problem

$$u_{xx} + u_{yy} = 0 \quad \text{in D},$$

$$u(0,y) = \begin{cases} 0 & y > 1 \\ 1 & y \leq 1 \end{cases}$$

$$u(x,0) = \begin{cases} 0 & x > 1 \\ 1 & x \leq 1 \end{cases}$$

and find the behavior of $u(x,y)$ far from the origin.

4. (a) Find the Green's function $G(x,y;\xi,\eta)$ given by

$$\nabla^2 G = \delta(x-\xi)\delta(y-\eta) \quad \text{in D},$$

$$\frac{\partial G}{\partial n} = 0 \quad \text{on } \partial\text{D},$$

$$G \sim \ln r \quad as\ r^2 = x^2 + y^2 \to \infty.$$

The domain D is the following region of the $x-y$ plane: $x \geq 0,\ y \geq 0,\ x^2+y^2 \geq 1$.

(b) How many images would be needed if the domain D changes to $D = \{r, \theta :\ r \geq 1,\ 0 \leq \theta \leq \frac{\pi}{4}\}$. Sketch the image system.

(c) An inviscid incompressible fluid occupies the region D in the $x-y$ plane defined by $x > 0,\ y > 0,\ x^2 + y^2 > 1$. The curved part of the boundary is porous and fluid is forced into D through it. If we denote the velocity <u>normal</u> to the curved boundary by v, then the boundary conditions at the wall are

$$v = \sin(2\theta) \quad \text{on} \ \ \mathrm{x}^2 + \mathrm{y}^2 = 1, \quad 0 \leq \theta \leq \frac{\pi}{2},$$

$$v = 0 \quad y = 0,\ x > 1,$$

$$v = 0 \quad x = 0,\ y > 1.$$

Write down the solution to this problem in terms of the streamfunction ψ and find an expression for the velocity on the line $\theta = \frac{\pi}{4}$.

5. The motion of a semi-infinite string is governed by the following set of equations:

$$(*) \qquad \phi_{xx} - \phi_{tt} = 0, \quad t > 0, \quad x > 0,$$

$$\phi(x,0) = \cos x, \qquad \phi_t(x,0) = 0, \qquad x > 0,$$
$$\phi(0,t) = e^{-t}, \qquad t > 0.$$

(a) Write down the general solution of the above partial differential equation $(*)$. (It will consist of two arbitrary functions, each being constant along a family of characteristics).

(b) Determine the solution of the above system in the region $x > t$. Note that the solution you construct will only be valid when the arguments of the two arbitrary functions are positive.

(c) Determine the solution of the above system in the region $x < t$. The boundary condition will enable you to determine the arbitrary function with negative argument in terms of the solution derived in (2).

6. A sphere of radius a is initially at a temperature T_0 and is surrounded by a fluid at temperature T_a. Here, $T_0 > T_a$. If the sphere looses heat to the fluid by convection, and this heat transfer is modeled by Newton's law of cooling

$$k\frac{\partial T}{\partial n} + h(T - T_a) = 0,$$

where k is the thermal conductivity of the sphere and h is the heat transfer coefficient assumed known, constant and independent of k. The sphere will cool because it is loosing energy to the surrounding fluid.

Prove or disprove that increasing k will increase the rate at which the sphere cools.

Hint: The Laplace operator in spherical polar coordinates r, θ, ϕ is given by

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$