Doctoral Qualifying Exam in Applied Mathematics. May 16, 2001.

You have three hours for this exam. Show all work in the books provided and attempt all questions. Complete answers are preferred to fragments.

1. Let

$$Lu = -\frac{d^2}{dx^2} \qquad x \in (0, L)$$

$$u(0) = 0, \qquad \frac{du}{dx}(L) = 0.$$

Find the Green's function $G(x, \xi, \lambda)$ for the operator $L - \lambda$. Use the Green's function and the result

$$\delta(x - \xi) = -\frac{1}{2\pi i}\int_{C_\infty} G(x, \xi, \lambda)\, d\lambda$$

to find the spectral decomposition of $\delta(x - \xi)$ and define the corresponding transform pair. Here C_∞ is a circle in the complex λ-plane with center at the origin and radius R in the limit $R \to \infty$.

2. Let

$$Lu = -\frac{d^2}{dx^2} \qquad x \in (0, \infty)$$

$$u(0) = 0, \qquad \int_0^\infty u^2(x)\, dx < \infty.$$

Find the Green's function $G(x, \xi, \lambda)$ for the operator $L - \lambda$. Use the Green's function and the result

$$\delta(x - \xi) = -\frac{1}{2\pi i}\int_{C_\infty} G(x, \xi, \lambda)\, d\lambda$$

to find the spectral decomposition of $\delta(x - \xi)$ and define the corresponding transform pair. Here C_∞ is a circle in the complex λ-plane with center at the origin and radius R in the limit $R \to \infty$.

3. Use the method of images to find the Green's function and the solution to the diffusion problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + f(x, t) \qquad x \in (0, \infty) \quad t > 0$$

$$u(x, 0) = 0, \qquad u(0, t) = 0$$

where $f(x, t)$ is given.

4. Use the Green's function and Poisson's formula to show that the solution to the Dirichlet problem for the interior of the unit circle, i.e.,

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \qquad r \in [0, 1) \quad \theta \in [0, 2\pi)$$

$$\text{with} \quad u(1,\theta) = f(\theta) \quad \text{given,}$$

is

$$u(r,\theta) = \frac{1-r^2}{2\pi} \int_0^{2\pi} \frac{f(\theta')\,d\theta'}{1+r^2-2r\cos(\theta-\theta')}.$$

Recall that the Green's function is

$$G(\vec{x},\vec{\xi}) = \frac{1}{2\pi} \ln \frac{|\vec{x}-\vec{\xi}|}{|\vec{\xi}||\vec{x}-\vec{\hat{\xi}}|}$$

where $\vec{x}$ and $\vec{\xi}$ are points inside the circle and $\vec{\hat{\xi}}$ is the inversion of $\vec{\xi}$.

Solve the same Dirichlet problem by separation of variables to show that

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} r^n (a_n \cos n\theta + b_n \sin n\theta),$$

and determine a_n and b_n in terms of $f(\theta)$. Show that this series can be found from the Poisson formula above when the integrand is expanded as a series in r.

5. Let $C : \mathbf{r} = x(t)\mathbf{i} + y(t)\mathbf{j}$ for $t \in [1,2]$ be a plane curve with $\mathbf{r}(1) = \mathbf{0}$ and $\mathbf{r}(2) = b\mathbf{i}$. Show that if

$$\frac{d}{dt}(F(|\mathbf{T}|)\hat{\mathbf{T}}) = p\frac{ds}{dt}\hat{\mathbf{N}},$$

where F is any differentiable function and p is a constant, then the curve is an arc of a circle. Here $\mathbf{T}$ is a tangent vector, $\hat{\mathbf{T}}$ is the unit tangent vector, $\hat{\mathbf{N}}$ is the unit normal, and s is the arc length on C.

6. Determine the effective spring constant for a system of two linear springs connected in series for small displacements about equilibrium. The springs have spring constants κ_1 and κ_2, the upper spring is suspended at its upper end from a rigid support, and the lower spring carries a point mass m at its lower end.