Ph.D Qualifying Exam in Applied Mathematics

May 12, 2000

Do all problems and carefully show all your work. Good luck!

Problem 1. A viscous incompressible fluid of kinematic viscosity ν occupies a region above an infinite flat plate. The flat plate moves in its own plane with a prescribed velocity $U\cos(\omega t)$.

Starting from the Navier-Stokes equations in two dimensions, find an appropriate solution for the flow in closed form. Specify clearly any assumptions you make.

Sketch the solution at a fixed time and discuss the physical significance of the parameter $(\omega/2\nu)^{1/2}$.

Problem 2. Consider the nonlinear pendulum problem

$$\frac{d^2\theta}{dt^2} + \theta - \alpha\theta^3 = 0, \quad \alpha > 0.$$

1. Find all equilibrium points and their stability.
2. If $\omega = \frac{d\theta}{dt}$, show that
$$\omega^2 + \theta^2 - \frac{1}{2}\alpha\theta^4 = E,$$
where E is a constant. Hence sketch the phase plane of the system and describe the dynamics.
3. If $\alpha \ll 1$, attempt to obtain a regular perturbation solution. What happens?
4. Fix the regular perturbation scheme above by using the method of multiple scales. Obtain the leading order term and in particular the equations which govern the evolution of the slowly varying amplitudes. (You do not need to solve these equations. You may need the identities $\sin 3x = 3\sin x - 4\sin^3 x$ and $\cos 3x = 4\cos^3 x - 3\cos x$.)

Problem 3. Let D denote the half space $\{(x,y,z) : z > 0\}$. Find the Green's function G satisfying

$$\begin{aligned}
\nabla^2 G + k^2 G &= \delta(x-x')\,\delta(y-y')\,\delta(z-z'), \quad (x,y,z),(x',y',z') \in D \\
\frac{\partial G}{\partial z} &= 0, \quad z = 0, |x| < \infty, |y| < \infty.
\end{aligned}$$

1. Solve the boundary value problem
$$\begin{aligned}
\nabla^2 u + k^2 u &= \delta(x)\delta(y)f(z), \quad (x,y,z) \in D \\
\frac{\partial u}{\partial x}(x,y,0) &= 0,
\end{aligned}$$
where f is given by
$$f(z) = \sin \pi z/L, \quad 0 < z < L, \quad f(z) = 0, \quad z > L.$$
2. Find the behavior of u as $r = (x^2+y^2+z^2)^{1/2} \to \infty$. This is a simple wire antenna model problem which arises in electromagnetics.

Problem 4. Consider the initial value problem

$$\begin{aligned} u_{tt} &= u_{xx} + f(x)g(t), \quad |x| < \infty, t > 0, \\ u &= u_t = 0, \quad t = 0, |x| < \infty, \quad u \text{ bounded as } |x| \to \infty. \end{aligned}$$

Here $f(x)$ is identically zero outside the bounded interval $x \in \Omega$, and $g(t) = H(t)\sin(\omega t)$ where $H(t)$ is the Heaviside function. Finally, let x_1 be a fixed point outside of Ω.

1. Show that $u(x_1, t) \equiv 0$ for $t \leq T_1$, where T_1 is the arrival time.
2. Show that $u(x_1, t) = v(x_1)\sin(\omega t)$ for $t \geq T_2$. That is, u is periodic after a finite amount of time.
3. Give geometrical interpretations of T_1 and T_2.

Problem 5. Solve the initial boundary value problem

$$\begin{aligned} u_y &= u_{xx}, \quad 0 < x < 1, t > 0 \\ u_x(0,t) &= 0, \quad u_x(1,t) + hu(1,t) = 0 \\ u(x,0) &= f(x), \end{aligned}$$

where $h \geq 0$ and $f(x)$ is a smooth function. Show that $u \to 0$ as $t \to \infty$ for all $0 < x < 1$ and that this decay becomes quicker as h is increased.

Problem 6. Let D denote the channel region $D := \{(x,y) : x > 0, 0 < y < 1\}$. Solve the boundary value problem

$$\begin{aligned} \nabla \cdot (y \nabla u) &= 0, \quad (x,y) \in D \\ u(0,y) &= f(y),\ 0 < y < 1, \quad u(x,1) = \frac{\partial u}{\partial y}(x,1) = 0,\ x > 0, \quad u \text{ bounded at } \infty, \end{aligned}$$

where $f(y)$ is a smooth function. This problem arises in hydrostatics.

[Hint: Consider the eigenvalues and eigenfunctions associated with the operator $\frac{d}{dy}\left(y\frac{d}{dy}\right)$ and an appropriate weight function.]