Doctoral qualifying exam:
Applied mathematics.

January 22, 1999.

You have three hours for this exam. Show all working in the books provided.

1. The one-dimensional relativistic motion of a particle of rest mass m_0 and velocity $v = \frac{dx}{dt}$ is governed by

$$\frac{d}{dt}\left(\frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}}\right) + kx = 0$$

where c is the speed of light and k is a positive constant. If a is the amplitude of an oscillation, so that $x = \pm a$ when $v = 0$, deduce the first integral of the motion

$$\frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{1}{2}kx^2 = m_0 c^2 + \frac{1}{2}ka^2.$$

Show that the oscillation has period

$$T = \frac{4}{c}\int_0^a \frac{f\,dx}{(f^2-1)^{1/2}}$$

where $f = 1 + \epsilon(a^2 - x^2)$ and $\epsilon = k/2m_0c^2$. Hence show that

$$T = 2\pi\sqrt{\frac{m_0}{k}}\;(1 + \frac{3}{8}\epsilon a^2 + O(\epsilon^2 a^4))$$

as $\epsilon a^2 \to 0$. What physical problem does this limit correspond to?

2. Consider the boundary value problem

$$u'' + u = f(x) \qquad x \in (0, \pi)$$

$$u(0) + \alpha u'(0) = c_1 \qquad u(\pi) = c_2$$

where the parameter α is real. For what values of α does a Green's function $G(x, \xi)$ exist? Construct the Green's function when it exists and give the solution of the boundary value problem $u(x)$ in terms of it. What condition must be satisfied by the data f, c_1, c_2 to ensure that a solution for u exists as α approaches values for which $G(x, \xi)$ fails to exist?

3. Show that the operator L defined by

$$Lu \equiv (1-x^2)^{1/2}\left(-(1-x^2)^{1/2}\,u'\right)' \qquad x \in (-1,1)$$
$$u(\pm 1) \text{ and } u'(\pm 1) \text{ bounded with } u(1) = 1,$$

where primes denote x-derivatives, is self-adjoint with respect to the inner product

$$< u, v > = \int_{-1}^{1} uv(1-x^2)^{-1/2}\,dx.$$

Show that $\lambda_0 = 0$ is an eigenvalue with eigenfunction $u_0 = 1$. Show that the operator L is positive definite for all other admissible functions $u \in \{C^1(-1,1)|u(\pm 1), u'(\pm 1) \text{ bounded}, u(1) = 1\}$. What can you conclude about the location of the eigenvalues λ_n in the complex plane and the properties of the eigenfuntions $u_n(x)$?

Explain why we can choose the eigensystem $(\lambda_n, u_n(x))$ to be such that

$$< u_m, u_n > = \begin{cases} 0 & m \neq n \\ \frac{\pi}{2} & m = n \neq 0 \\ \pi & m = n = 0. \end{cases}$$

Given that $\lambda_1 = 1$ and $u_1 = x$, and that u_2 is an even polynomial of degree two, find or approximate λ_2 and u_2. How would you find or approximate other eigenvalues and eigenfunctions?

4. (i) Consider the boundary value problem

$$\nabla^2 \psi = 0$$
$$\psi = 0 \text{ on } y = 0,\ |x| > a$$
$$\psi = 0 \text{ on } x^2 + y^2 = a^2,\ y > 0$$
$$\psi \sim cy \text{ as } x^2 + y^2 \to \infty.$$

Put $\chi = \psi - cy$ to find the problem for χ. Find the Green's function $G(\vec{x}, \vec{\xi})$, and hence find the solution for ψ in terms of $G(\vec{x}, \vec{\xi})$. Can you simplify this expression?

(ii) Let the half-circle radius a tend to zero in the boundary value problem of (i), so that ψ is the solution of

$$\nabla^2 \psi = 0 \qquad x \in (-\infty, \infty),\ y > 0$$
$$\psi = 0 \text{ on } y = 0,\ \psi \sim cy \text{ as } x^2 + y^2 \to \infty.$$

Put $\chi = \psi - cy$ to find the boundary value problem for χ and then write down the solution for ψ. Find the conformal map that maps the half-space $y > 0$ in the z-plane to the wedge with angle α (i.e., $r > 0$, $\theta \in (0, \alpha)$) in the w-plane, and hence find the potential $\Psi(r, \theta)$ in the w-plane. Find the values of c and α such that $\Psi(r, \theta)$ satisfies the boundary condition

$$\left(\frac{1}{r}\frac{\partial \Psi}{\partial \theta}\right)^2 - 2r\cos\frac{\alpha}{2} = \text{constant}$$

on the boundaries of the wedge $\theta = 0, \alpha$.

5. For the boundary value problem

$$\begin{aligned} u_{tt} - u_{xx} &= p(x,t) \qquad x \in (0,1), \quad t > 0 \\ u(x,0) &= f(x), \qquad u_t(x,0) = g(x), \\ u(0,t) &= l(t), \qquad u_x(1,t) = m(t), \end{aligned}$$

find the Green's function $G(x,\xi,t,\tau)$ by the method of images, and sketch G with its image system in the x,t-plane. Find the eigenfunction expansion of the Green's function. Find the representation of the solution for $u(x,t)$ in terms of the Green's function.

6. A frozen chicken is moved from the freezing compartment of a refrigerator to the main part. It is found that it takes approximately 10 hours for it to thaw. Make a simple model for the problem and use it to estimate the thermal conductivity of the chicken.