Doctoral Qualifying Exam:
Applied Mathematics.

Wednesday, January 21, 1998.

You have three hours for this exam. Show all working in the books provided.

1. Solve the following initial value problem by constructing an appropriate Green's function.

$$(x+1)^2 u'' + 2(x+1)u' = f(x), \qquad 0 < x < \infty,$$

$$u(0) = 1, \qquad u'(0) = 1.$$

2. Solve the following boundary value problem by constructing the Green's function (or a modified Green's function, if necessary). Indicate any consistency condition that might be required for solvability.

$$u'' - 2u' + u = f(x), \qquad 0 < x < 1,$$

$$u(0) = u(1), \qquad u'(0) = 0.$$

3. The temperature of a semi-infinite rod satisfies

$$u_t - u_{xx} = 0 \quad x \in (0,\infty),\ t > 0$$

$$u(x,0) = f(x), \quad u(0,t) = 0.$$

Find the Green's function for this problem and give the solution for u in terms of it. Show that for constant initial data $f(x) = 1$, the solution can be simplified to $u(x,t) = \mathrm{erf}\left(\frac{x}{2\sqrt{t}}\right)$, where $\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-s^2}\,ds$ is the error function.

4. (a) A potential Φ is the solution of the Dirichlet problem

$$\nabla^2\Phi = 0 \quad x \in (-\infty,\infty),\ y > 0\ , \qquad \Phi(x,0) = \begin{cases} 1 & x \in (0,1) \\ -1 & x \in (-1,0) \\ 0 & \text{elsewhere} \end{cases}$$

Find the Green's function for this problem and give the solution for Φ in terms of it.
(b) Use the solution for Φ and an appropriate conformal map to find the solution ϕ of the Dirichlet problem

$$\nabla^2\phi = 0 \quad r > 0,\ \ \theta \in (0, \frac{2\pi}{3})\ , \qquad \phi(r,\theta) = \begin{cases} 1 & r \in (0,1),\ \theta = 0 \\ -1 & r \in (0,1),\ \theta = \frac{2\pi}{3} \\ 0 & r > 1,\ \theta = 0 \text{ and } \theta = \frac{2\pi}{3} \end{cases}$$

Note: You may be able to simplify your answers by using the formula

$$\arctan A \pm \arctan B = \arctan\left(\frac{A \pm B}{1 \mp AB}\right)$$

5. Consider the boundary value problem

$$\nabla^2 U + k^2 U = F(x, z), \qquad (x, z) \in D$$

$$U(0, z) = U(1, z) = 0, \quad 0 < z < \infty, \qquad U(x, 0) = 0, \quad 0 < x < 1,$$

where $D = \{(x, z) | \, 0 < x < 1, 0 < z < \infty\}$ and $k \in \mathbf{R}$ is given. The forcing F has compact support, that is, F is identically zero outside a compact region Ω which is completely contained in D. The solution U also satisfies the boundary condition that it represent outgoing waves as $z \to \infty$.
Construct the Green's function for this problem and use it to express the solution of the boundary value problem. Show that for z sufficiently large

$$U \sim \sum_{n=1}^{M} T_n \sin n\pi x \, e^{ik_n z}$$

where $k_n = \sqrt{k^2 - n^2\pi^2} \in \mathbf{R}$ for $n \leq M$, and express the coefficient T_n in terms of F.

6. A hot, infinite cylindrical rod at temperature T_0 is completely submerged in a cool bath which is maintained at temperature $T_i < T_0$. Show that if the diameter of the rod is doubled then the time taken for its center to cool to a temperature $T_c = (T_0 + T_i)/2$ increases by a factor of four. Find an equation determining the time taken to cool to the temperature T_c. (The heat equation with axial symmetry is $\frac{\partial T}{\partial t} = \kappa \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial T}{\partial r}\right)$, and the solution of $z^2 w'' + z w' + \lambda z^2 w = 0$ which is bounded at $z = 0$ is the Bessel function $w = J_0(\sqrt{\lambda} z)$ with $J_0(0) = 1$.)