Applied Mathematics Qualifying Examination: Feb. 5, 1997

Do all the problems and carefully show all of your work. Good luck!

1.) Consider the eigenvalue problem

$$\frac{d}{dx}(p(x)\frac{d}{dx}u) + [\lambda r(x) - q(x)]u = 0, \quad 0 < x < 1$$

$$u(0) = u(1) = 0, \quad x = 0, 1$$

where $p(x)$,$r(x)$, and $q(x)$ are positive and smooth on $0 < x < 1$.

Let $G(x, x')$ be the Green's function of the related problem

$$\frac{d}{dx}(p(x)\frac{d}{dx}G) - q(x)G = \delta(x - x'), \quad 0 < x, x' < 1$$

$$G(0, x') = G(1, x') = 0.$$

a) Show that:

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = \int_0^1 r(x)G(x, x)\, dx.$$

b) Show also that:

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} = \int_0^1 \int_0^1 r(x)r(y)G(x, y)G(y, x)\, dx\, dy.$$

c) Use the result in (a) to sum the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2\pi^2}.$$

[Hint: Choose a particular p,r, and q].

2.) Consider the same eigenvalue problem as in question (1) and let λ_1 be the smallest eigenvalue. Consider the functional $I : D \to \Re$ defined by

$$I(u) = \int_0^1 [p(x)(\frac{d}{dx}u)^2 + q(x)u^2]\, dx$$

where $D = \{u(x)|u(0) = u(1) = 0,\ u(x) \text{is smooth on} \, 0 < x < 1\}$.

a.) Carefully state the relationship between $I(u)$, D, and λ_1.

b.) Let $\phi_1(x) = \sin(\pi x)$. Use the Rayleigh-Ritz method and this function to construct an approximation $\tilde{\lambda_1}$ to λ_1. Is $\tilde{\lambda_1}$ bigger or less than λ_1?

3.) For the initial boundary value problem

$$u_t - u_{xx} = p(x,t), \qquad x \in (0,1),\ t > 0$$

$$u(x, 0^+) = f(x)$$

$$u_x(0,t) = g(t), \qquad u_x(1,t) = h(t),$$

use the method of images to find the Green's function G in terms of the fundamental solution, then express the solution for u in terms of G. Solve the same problem by eigenfunction expansion, and use this to identify the eigenfunction expansion of G.

Approximate solutions for u are given by truncating each of these two representations of G. Explain when you expect each approximation to be a good approximation to u, and how well each approximation satisfies the boundary conditions.

Find a necessary condition for the total thermal energy $Q(t) = \int_0^1 u(x,t)\, dx$ to approach a constant as $t \to \infty$. Also find necessary conditions for the solution $u(x,t)$ to approach a steady state $u_\infty(x)$ as $t \to \infty$.

4.) A sphere of radius R is initially at temperature T_i and is placed in an oven where the temperature is maintained at a constant temperature $T_0 > T_i$. The sphere is to be removed when its center reaches T_E, where $T_0 > T_E > T_i$. Show that the time required to "cook" the sphere is proportional to $(Volume)^{2/3}$.

How does the "cooking" time vary with the thermal conductivity? [Hint: $\nabla^2 = \dfrac{\partial^2}{\partial r^2} + \dfrac{2}{r}\dfrac{\partial}{\partial r}$.]