Applied Mathematics Qualifying Examination
January 1996

1. (a) If q^2 is a positive constant, find the Green's function $G(x, \xi)$ for the following problem, which has a source placed at $x = \xi$:

$$-\frac{d^2G}{dx^2} + q^2 G = \delta(x - \xi), \quad 0 < x, \xi < 1, \quad \frac{dG}{dx}(0, \xi) = \frac{dG}{dx}(1, \xi) = 0 \tag{1}$$

What happens to your solution as $q^2 \to 0$?
(Show explicitly in your calculations the drivation of any jump conditions you may need to use.)
(b) Show that when $q^2 = 0$ problem (1) above does not have a solution. Give a physical explanation by interpreting the problem in terms of one-dimensional heat flow.

2. Let the operator L be defined by

$$Lu \equiv -\frac{1}{x}(xu')',$$

with boundary conditions

$$|u(0)| < \infty, \qquad u(1) = 0,$$

where primes denote x-derivatives.
(a) If the scalar product is defined by

$$< u, v > = \int_0^1 xu(x)v(x)dx$$

show that L is self-adjoint and positive definite. What can be said about the sign of the eigenvalues of L?
(b) By considering the eigenvalue problem

$$Lu = \lambda u$$

use the Rayleigh-Ritz theorem and an appropriate quadratic polynomial trial function to find an approximation to the first zero of $J_0(x)$, where J_0 is the Bessel function of the first kind and order zero which is bounded at the origin. Give a justification for your choice of trial function. You can quote the following facts about Bessel functions: the equation $(xu')' + xu = 0$ has independent solutions J_0 and Y_0. The solution J_0 is bounded at $x = 0$ while Y_0 is not, with $J_0(0) = 1$ and $J_0'(0) = 0$.

3 (a) Use the method of images to solve the following Green's function problem:

$$\nabla^2 G = \delta(x - \xi)\delta(y - \eta) \quad x, \xi \in (-\infty, \infty), \quad y, \eta > 0$$
$$\frac{\partial G}{\partial y} = 0 \;\text{ on } y = 0, \; x \in (-\infty, \infty)$$
$$G \sim \ln r \text{ as } r^2 = x^2 + y^2 \to \infty.$$

(b) The inviscid flow above a wing which has aerodynamic control via suction and blowing strips on its surface can be described by the following problem for the fluid velocity potential ϕ:

$$\nabla^2\phi = 0 \quad x \in (-\infty,\infty), \quad y > 0$$

$$\text{on } y=0: \quad \frac{\partial \phi}{\partial y} = \begin{cases} \alpha & x \in (1,2) \\ \beta & x \in (-2,-1) \\ 0 & \text{elsewhere} \end{cases}$$

$$\phi \sim x \text{ as } x^2+y^2 \to \infty.$$

The constants α and β can be positive (blowing) or negative (suction). Use the Green's function found in (a) to write down an integral representation of the solution.

The velocity field is given by $(u,v) = \nabla\phi$. Show that

$$u(\xi,\eta) = 1 + \frac{1}{2\pi}\ln\left(\frac{((1+\xi)^2+\eta^2)^\alpha((2-\xi)^2+\eta^2)^\beta}{((1-\xi)^2+\eta^2)^\beta((2+\xi)^2+\eta^2)^\alpha}\right)$$

$$v(\xi,\eta) = \frac{\alpha}{\pi}\left[\tan^{-1}(\frac{1+\xi}{\eta}) - \tan^{-1}(\frac{2+\xi}{\eta})\right] + \frac{\beta}{\pi}\left[\tan^{-1}(\frac{1-\xi}{\eta}) - \tan^{-1}(\frac{2-\xi}{\eta})\right].$$

Use these solutions to show that the boundary data is attained as $\eta \to 0^+$. Show that the flow will reverse direction at the origin if α and β satisfy the inequality

$$\alpha > \beta + \frac{2\pi}{\ln 2}.$$

4. Consider the telegraph equation

$$u_{tt} - u_{xx} + au_t + bu = 0 \qquad x \in (-\infty,\infty), \quad t > 0 \tag{2}$$

where $a > 0$. Find a function $\omega(t)$ and a constant k (in terms of a and b) such that $u(x,t) = \omega(t)v(x,t)$ where $v(x,t)$ satisfies the equation

$$v_{tt} - v_{xx} + kv = 0.$$

(a) Show that for the special case $b = -\frac{a^2}{4}$ the constant k vanishes, and find the general solution of (2) in this case.

(b) Describe the solution you found in (a) when the following initial conditions are used

$$u(x,0) = \begin{cases} 0 & |x| > 1 \\ 1 & -1 \le x \le 1 \end{cases}, \qquad u_t(x,0) = 0.$$

5. (a) The integral form of the conservation law for a density u and its flux $F(u)$ is

$$\frac{d}{dt}\int_{x_1}^{x_2} u\,dx + F(u)|_{x_2} - F(u)|_{x_1} = 0.$$

Show that if the solution $u(x,t)$ is discontinuous across a path $x = x_s(t)$ then the path of the discontinuity satisfies

$$\frac{dx_s}{dt} = \frac{[F(u)]}{[u]}$$

where $[\cdot]$ denotes the jump across the discontinuity.

(b) When $u(x,t)$ is differentiable the integral form of the conservation law can be replaced by a partial differential equation. Write down the PDE and then set $F(u) = \phi(u) - \nu u_x$ to find

$$u_t + \phi'(u)u_x = \nu u_{xx}. \tag{3}$$

When $\nu > 0$ what type of physical effect does the term in ν represent and what might its physical origins be?

Nondimensionalize (3) by putting

$$x = x_*\tilde{x}, \quad t = t_*\tilde{t}, \quad u = u_*\tilde{u}, \quad \frac{d\phi(u)}{du} = \alpha_*\tilde{f}(\tilde{u}), \quad \epsilon = \frac{\nu t_*}{x_*^2},$$

where a tilde denotes a dimensionless variable and an asterisk denotes a dimensional scale. Show that, after dropping the tildes, a suitable choice of α_* gives the nondimensional equation

$$u_t + f(u)u_x = \epsilon u_{xx}$$

What choice of α_* was made and, for an initial value problem, what would x_* and t_* represent?