Applied Mathematics Qualifying Examination
January 1995

1. Find all real eigenvalues λ of the following problem

$$x^2y'' - \lambda xy' + \lambda y = 0,$$
$$y(1) = 0, \qquad y(2) - y'(2) = 0.$$

What are the corresponding eigenfunctions?

2. The nonlinear Burgers equation is given by

$$u_t + uu_x = \mu u_{xx} \tag{1}$$

$-\infty < x < \infty,\ t > 0$ where $\mu > 0$. Show that the function $v(x,t)$ given by

$$v(x,t) = -\frac{2\mu u_x}{u},$$

satisfies

$$v_t = \mu v_{xx}$$

Use this fact and the free-space Green's function for the heat equation to find the solution of (1) subject to

$$u(x,0) = f(x), \qquad \lim_{t\to\infty} u(x,t) = 0.$$

3. Consider a semi-infinite transmission line starting at $x = 0$. The voltage across the line $V(x,t)$ and the current along the line $U(x,t)$ are related by the following normalized system of partial differential equations

$$V_t = -U_x$$
$$U_t = -V_x$$

on $x > 0, t > 0$. Initially $V(x,0) = f(x)$, where $f(x)$ is non-zero on the bouded interval $x \in [a,b]$ and $f(x) = 0$ elsewhere, and $U(x,0) = 0$. Find a solution for $V(x,t)$. Sketch the voltage at $x = 0$ as a function of t if $f(x) = 1$ for $x \in [a,b]$.

4. Construct approximate solutions for $0 < \epsilon \ll 1$ to the initial value problem for the viscous Burgers equation

$$\begin{aligned} u_t + f(u)u_x &= \epsilon u_{xx} \\ u(x,0) &= g(x) \end{aligned} \tag{2}$$

where the initial data $g(x)$ is smooth, as follows:

(a) Look for the leading term of an outer expansion by setting $\epsilon = 0$. Write down the solution of the reduced equation as given by the method of characteristics. Briefly explain the circumstances under which this solution 'breaks', and find general expressions for the values of x and t at which breaking occurs.

(b) When $\epsilon > 0$ the solution must remain smooth for all $t > 0$. Look for the leading term of an inner solution, which can be used to replace the outer solution when it is multi-valued by a solution that is single-valued but has a narrow transition layer. To do this, change variables from x to ξ by putting

$$x = x_s(t) + \epsilon\xi \tag{3}$$

where ξ is a local distance coordinate centered on the layer and is independent of t. Find the exact form of (2) that is given by the change of coordinates (3). Hence show that, at leading order for small ϵ, the solution $u = U(\xi, t)$ in the layer satisfies

$$(f(U) - \frac{dx_s}{dt})U_\xi = U_{\xi\xi} \tag{4}$$

Give the boundary conditions that $U(\xi, t)$ must satisfy as $\xi \to \pm\infty$. Integrate equation (4) and apply the boundary conditions to find a relation that must be satisfied by the speed of the moving layer. How does this compare to the expression for the speed of propagation of a discontinuity or shock?