Ph.D Qualifying Exam in Applied Mathematics

January 18, 2002.

PROBLEM 1. An object of initial mass M_0 travels at a constant speed V_0 and at $t = 0$ it enters a region (of infinite extent in the direction of motion) which contains a viscous fluid. It is observed that the viscous fluid has two significant effects on the motion:

- It causes a frictional *retarding* force on the object which is proportional to its instantaneous velocity. To fix things introduce a constant of proportionality $k > 0$.
- The fluid causes the solid to dissolve such that the mass of the solid *decreases* at a constant rate. To fix things, introduce a constant of proportionality $\alpha > 0$.

Construct a mathematical model of the above problem and state the dimensions of k and α in your model.

(a) Show that the speed of the object at any instant is given by

$$v(t) = \left(\frac{M_0 - \alpha t}{M_0}\right)^{\frac{k-\alpha}{\alpha}} V_0.$$

(b) Show that if $k > \alpha$, the particle comes to rest after a finite time. Show also that this happens at a finite distance from the point of entry into the fluid region and find this distance.

(c) What happens when $k = \alpha$ and $k < \alpha$? For the latter case, show that the total distance covered by the particle remains finite and equal to $M_0 V_0 / k$.

(d) Sketch the position of the particle as a function of t when $k > \alpha$ and $k < \alpha$.

PROBLEM 2. Consider a self adjoint system $L[u] = f(x)$ on $0 \le x \le 1$, with linear homogeneous boundary conditions at $x = 0, 1$, e.g. $u(0) = u(1) = 0$. Let λ_n be the eigenvalues of $L[u] = \lambda u$, and let u_n be the corresponding eigenvectors. Suppose $\lambda = 0$ is not an eigenvalue of the system.

(a) Show that the solution of this system is

$$u(x) = \sum_1^\infty \frac{a_n u_n(x)}{\lambda_n},$$

where $a_n =< u_n, f >$.

(b) Comparing the above expansion with the Green's function representation, deduce that

$$G(x,y) = \sum_1^\infty \frac{u_n(x)\ u_n(y)}{\lambda_n}$$

(Assume the order of integration and summation can be interchanged where necessary.)

(c) Applying L to the above, deduce at least formally that

$$\sum_1^\infty u_n(x) u_n(y) = \delta(x - y).$$

Show that this is true as equality of distributions on $0 < x < 1$ (use the fact that a test function $\phi \in C_0^\infty(0,1)$ will have a uniformly convergent eigenfunction expansion).

(d) Deduce that

$$\int_0^1 G(x,x)\ dx = \sum_1^\infty \frac{1}{\lambda_n}.$$

PROBLEM 3. Consider the boundary value problem

$$-\frac{d^2y}{dx^2} = f(x); \quad y(0) = y(l), \quad y'(0) = y'(l)$$

modeling steady-state heat conduction in an insulated thin *ring*.

(a) What is the adjoint problem?

(b) What condition on $f(x)$ is required for the existence of a solution?

(c) Find the modified Green's function for the problem. Write the solution in terms of the Green's function.

PROBLEM 4. We need to find the two-dimensional steady temperature distribution in the first quadrant when the vertical wall is held at zero temperature and the horizontal one is insulated. In addition, a heat source of constant value σ_0 per unit area acts on the annulus $a \leq r \leq a+b$, $0 < \theta < \pi/2$, where polar coordinates have been introduced in the usual way.

(a) Write down the problem to be solved. Include all boundary conditions.

(b) Find the Green's function for this problem.

(c) Use the Green's function to obtain a formula for the value of the temperature at any point in the first quadrant.

(d) Find the leading behavior of the solution at large distances from the origin.

PROBLEM 5. Find the solution of the problem

$$\begin{aligned} u_t - u_{xx} &= p(x,t) \qquad 0 \leq x \leq 1, \quad 0 \leq t \\ u(x, 0+) &= f(x) \\ u(0,t) &= 0, \quad u_x(1,t) = h(t). \end{aligned}$$

PROBLEM 6. Solve the following problem:

$$\begin{aligned} u_{tt} &= u_{xx} \qquad x > 0,\ t > 0, \\ u(0,t) &= y(t), \\ u(x,0) &= 0, \quad u_t(x,0) = 0. \end{aligned}$$

where the function $y(t)$ satisfies the differential equation

$$\begin{aligned} y_{tt} + y &= u_x(0,t), \\ y(0) = 1, &\quad y_t(0) = 0. \end{aligned}$$

This problem models radiation damping of oscillations by a semi-infinite string. It was originally posed by Rayleigh.