Doctoral qualifying exam:
Applied mathematics.

January, 2001.

Choose six out of the following seven questions. You have three hours for this exam. Show all working in the books provided.

1. The relativistic one-dimensional motion of a particle of rest mass m_0 and velocity $v = \frac{dx}{dt}$ is governed by

$$\frac{d}{dt}\left(\frac{m_0 v}{\sqrt{1-\frac{v^2}{c^2}}}\right) + kx = 0$$

where c is the speed of light and k is a positive constant. If a is the amplitude of an oscillation, so that $x = \pm a$ when $v = 0$, deduce the first integral of the motion

$$\frac{m_0 c^2}{\sqrt{1-\frac{v^2}{c^2}}} + \frac{1}{2}kx^2 = m_0 c^2 + \frac{1}{2}ka^2.$$

Show that the oscillation has period

$$T = \frac{4}{c}\int_0^a \frac{f\,dx}{(f^2-1)^{1/2}}$$

where $f = 1 + \epsilon(a^2 - x^2)$ and $\epsilon = k/2m_0c^2$. Hence show that

$$T = 2\pi\sqrt{\frac{m_0}{k}}\,(1 + \frac{3}{8}\epsilon a^2 + O(\epsilon^2 a^4))$$

as $\epsilon a^2 \to 0$. What physical problem does this limit correspond to?

2. A sphere of radius R is initially at temperature T_i and is placed in an oven where the temperature is maintained at a constant temperature $T_0 > T_i$. The sphere is to be removed when its center reaches a given temperature T_E, where $T_0 > T_E > T_i$. Show that the time required to 'cook' the sphere so that its temperature reaches T_E is proportional to $(\textit{sphere volume})^{2/3}$.

How does the cooking time vary with thermal conductivity? (Hint: $\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}$.)

3. Consider the boundary value problem

$$u'' - 2u' + u = f(x) \qquad x \in (0,1)$$

$$u'(0) + \alpha u(0) = c_1 \qquad u(1) = c_2$$

where the parameter α is real. For what values of α does a Green's function $G(x,\xi)$ exist? Construct the Green's function when it exists and give the solution of the boundary value problem $u(x)$ in terms of it. What condition must be satisfied by the data f, c_1, c_2 to ensure that the solution for u remains bounded as $\alpha \to 0$?

4. Show that the operator L defined by

$$Lu \equiv (1-x^2)^{1/2}\left(-(1-x^2)^{1/2}\,u'\right)' \qquad x \in (-1,1)$$
$$u(\pm 1) \text{ and } u'(\pm 1) \text{ bounded with } u(1)=1,$$

where primes denote x-derivatives, is self-adjoint with respect to the inner product

$$< u, v >= \int_{-1}^{1} uv(1-x^2)^{-1/2}\, dx.$$

Show that $\lambda_0 = 0$ is an eigenvalue with eigenfunction $u_0 = 1$. Show that the operator L is positive definite for all other admissible functions $u \in \{C^1(-1,1)|u(\pm 1), u'(\pm 1) \text{ bounded}, u(1)=1\}$. What can you conclude about the location of the eigenvalues λ_n in the complex plane and the properties of the eigenfunctions $u_n(x)$?

Explain why we can choose the eigensystem $(\lambda_n, u_n(x))$ to be such that

$$< u_m, u_n >= \begin{cases} 0 & m \neq n \\ \frac{\pi}{2} & m = n \neq 0 \\ \pi & m = n = 0. \end{cases}$$

Given that $\lambda_1 = 1$ and $u_1 = x$, and that u_2 is an even polynomial of degree two, find or approximate λ_2 and u_2. How would you find or approximate other eigenvalues and eigenfunctions?

5. The Neumann problem for Laplace's equation in a half-space is

$$\nabla^2 u = 0 \quad x \in (-\infty, \infty) \quad y \in (-\infty,\infty) \quad z > 0$$

$$\frac{\partial u}{\partial z}(x,y,0) = \begin{cases} f(x,y) & x^2+y^2 < 1 \\ 0 & x^2+y^2 \geq 1. \end{cases}$$

Find the Green's function for this problem and give the solution for u in terms of it.

Show that far from the origin, as $|\mathbf{x}| \to \infty$, the solution for u has the expansion

$$u(\mathbf{x}) \sim -\frac{1}{2\pi\,|\mathbf{x}|}\int\!\!\int_{\Omega} f(\xi,\eta)\, d\xi d\eta - \frac{1}{2\pi\,|\mathbf{x}|^3}\int\!\!\int_{\Omega} f(\xi,\eta)\,\mathbf{x}\cdot\xi_0\, d\xi d\eta + O(|\mathbf{x}|^{-3})$$

where $\mathbf{x} = (x,y,z)$, $\xi_0 = (\xi,\eta,0)$, and Ω is the interior of the unit circle. Explain why the second term in the expansion is zero when f is constant.

6. Write down the Green's function and the solution to the initial boundary value problem

$$u_t - u_{xx} = p(x,t) \quad x \in (0,\infty) \quad t > 0$$
$$u_x(0,t) = g(t) \quad t > 0, \quad u(x,0) = f(x) \quad x > 0.$$

Show that the contribution to the solution from the boundary data alone can be written

$$u = \frac{-1}{\sqrt{\pi}}\int_0^t g(t-\tau)\frac{e^{-\frac{x^2}{4\tau}}}{\tau^{\frac{1}{2}}}\, d\tau.$$

7. Let $D = \{(x, y, z) \mid 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1\}$ be a 'cubical resonator' and let ∂D denote its sides.

(a) Find the Green's function satisfying

$$\begin{aligned} G_{tt} &= \nabla^2 G + \delta(\mathbf{x} - \mathbf{x_0})\delta(t - t_0), \quad \mathbf{x} \in D \\ G &= G_t = 0,\ t < t_0 \\ G &= 0, \quad \mathbf{x} \in \partial D \end{aligned}$$

where $\mathbf{x}$ and $\mathbf{x_0}$ are in D.

(b) Solve the initial-boundary value problem

$$\begin{aligned} u_{tt} &= \nabla^2 u \quad \mathbf{x} \in D,\ t > 0 \\ u &= u_t = 0, \quad t = 0 \\ u(x, y, 0, t) &= f(x, y)\sin(\nu t), \quad (x, y) \in \Omega \\ u &= 0, \quad \mathbf{x} \in \partial D - \Omega \end{aligned}$$

where Ω is a simply connected region on the face $z = 0$ of the cube.