Applied Mathematics Qualifying Examination: Jan. 14, 2000

Do all the problems and carefully show all of your work. Good luck!

1. Consider the following boundary value problem

$$\frac{1}{R}(u_{xx}+u_{yy}) - u_{xxxx} - u_t - u_{xx}u_y = 0,$$

where $u = u(x,y,t)$, $-\infty < x < \infty$, $t > 0$, and $0 < y < 1$ with

$$u(x,0,t) = 0, \qquad u(x,1,t) = 1.$$

(a) Letting $u = y + \tilde{u}(x,y,t)$, where $u_0 = y$ is an equilibrium solution, determine the boundary conditions and the linearized perturbation equations for $\tilde{u}$.

(b) Assume a solution of the form $\tilde{u} = f(y)\exp(\sigma t)\cos kx$ and show that the fundamental (normal) modes are

$$\tilde{u}_n = \exp(\sigma_n t)\cos kx \sin n\pi y$$

where

$$\sigma_n = k^2 - k^4 - \frac{k^2}{R} - \frac{n^2\pi^2}{R}, \qquad n = 1,2,\ldots$$

(c) Show that if R exceeds $(k^2+n^2\pi^2)(k^2-k^4)^{-1}$, then the nth mode grows without bound.

(d) Assuming perturbations of all wave numbers $0 < k < 1$ are present, at what value of R will instability set in?

2. Consider a saturated liquid at freezing temperature T_f of infinite extent outside a sphere of radius R_w. At $t = 0$, the surface at R_w is suddenly lowered to a subfreezing temperature T_0. Write down a suitable model, and show that it can be nondimensionalized as

$$\theta_t = \frac{1}{r}\frac{\partial^2}{\partial r^2}(r\theta),$$

$$\theta = 0 \quad \text{on} \quad r = 1,$$

$$\theta = 1, \quad \frac{\partial \theta}{\partial r} = St\ \dot{R} \quad on \quad r = R(t),$$

where the Stefan number is $St = L/c_p(T_f - T_0)$.

If $St \gg 1$, define $\epsilon = 1/St$, $t = (St)\tau$, and show that the freezing front $R(\tau)$ is described by

$$\tau = \frac{2R^3 - 3R^2 + 1}{6} + \frac{1}{6}\epsilon(R-1)^2 - \frac{1}{45}\epsilon^2\frac{(R-1)^2}{R} + O(\epsilon^3).$$

3. Consider the eigenvalue problem

$$\frac{d}{dx}\left(p(x)\frac{d}{dx}u\right) + [\lambda r(x) - q(x)]u = 0, \quad 0 < x < 1 \tag{1}$$

$$u(0) = u(1) = 0.$$

where $p(x)$,$r(x)$, and $q(x)$ are positive and smooth on $0 < x < 1$.

Let $G(x,x')$ be the Green's function of the related problem, i.e.,

$$\frac{d}{dx}\left(p(x)\frac{d}{dx}G\right) - q(x)G = \delta(x - x'), \quad 0 < x, x' < 1$$

$$G(0,x') = G(1,x') = 0.$$

Let $\{\lambda_n\}$ be the eigenvalues of (1).

(a) Show that:

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n} = -\int_0^1 r(x)G(x,x)dx.$$

(b) Show also that:

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} = \int_0^1 \int_0^1 r(x)r(y)G(x,y)G(y,x)\ dx\ dy.$$

(c) Use the result in (a) to sum the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2\pi^2}.$$

[Hint: Choose a particular p,r, and q].

4. Let D denote the half space $\{(x,y,z)|z>0\}$.

(a) Find the Green's function G satisfying

$$\nabla^2 G + k^2 G = \delta(x-x')\delta(y-y')\delta(z-z'), \quad (x,y,z)\in D, \quad (x',y',z')\in D$$

$$G=0, \quad z=0, \quad |x|<\infty, \quad |y|<\infty.$$

(b) Solve the boundary value problem

$$\nabla^2 u + k^2 u = 0, \quad (x,y,z)\in D$$

$$u(x,y,0) = f(x,y)$$

where f is zero outside a compact region R.

(c) Find the behavior of u as $r\to\infty$. This is an aperture problem which arises in acoustics and electromagnetics.

5. Let D denote the interior of the unit circle. Solve the initial boundary value problem

$$u_t = \nabla^2 u + p(t)\delta(r-r_0), \quad (x,y)\in D$$

$$u(x,y,0)=0, \qquad (x,y)\in D$$

$$u(x,y,t)=0 \qquad x^2+y^2=1$$

where $0<r_0<1$ and $p(t)$ is a smooth function on $t>0$ and identically zero for $t<0$. The source in this equation is a model of a wire-ring heater.

6. Let D denote the interior of the ellipse $x^2+\frac{y^2}{3}=1$ and let $p(x,y)=1+\sin^2 xy$. Find a solution to the boundary value problem

$$\nabla\cdot(p\nabla u)=0, \quad (x,y)\in D$$

with the mixed boundary conditions $u=1$ on the upper half of the ellipse and $\frac{\partial u}{\partial n}=0$ on the lower half where $\frac{\partial}{\partial n}$ denotes the normal derivative. Is your solution unique?