Applied Mathematics PhD Qualifying Examination

January 15-16, 2004

1. Data from the Hudson Bay Company in Canada goes back to 1840 and shows periodic cycles in the populations of snowshoe hare (prey - P) and lynx (predator - Q).

   (a) Derive a model for these two populations, P and Q, under the assumptions:

      i. the prey population, P, grows exponentially in time if there are no predators,

      ii. the rate of predation, i.e., the rate at which the predators eat the prey, depends on the likelihood of encounters between the two populations, P and Q,

      iii. the growth rate of the predator population is proportional to food intake, i.e., proportional to the predation rate,

      iv. the predator population dies off exponentially in time if there are no prey.

   (b) Nondimensionalize the variables and parameters in these two equations and determine any remaining dimensionless parameter(s). Give an interpretation of the dimensionless parameter(s).

   (c) Determine the steady states of the system and their stability. Classify the steady states.

   (d) Sketch the (P, Q) phase plane including the nullclines and steady states.

   (e) Sketch the direction fields through the nullclines and of a few trajectories in the phase plane.

   (f) Determine the period of small amplitude oscillations about the steady state.

   (g) What can you say about the structural stability of this system?

2. Consider the following problem

$$-u_{xx} - \lambda u = f(x), \quad 0 \leq x \leq l, \quad u(0) = u(l) = 0\,,$$

   where λ is a given complex number and $f(x)$ is a given square integrable complex function.

   (a) Write down the solution for u. What is the uniqueness condition?

   (b) Find the Green's function for this problem (put $f(x) = \delta(x - \xi)$) in terms of the eigenfunctions of the corresponding homogeneous problem.

3. Find the consistency condition that must be satisfied in order for the following boundary value problem to have a solution

$$u'' + \pi^2 u = f(x); \quad u(0) = \alpha; \quad u'(1/2) = \beta\,,$$

   where α, β are constants, and f is a given well-behaved function.

4. Consider a 1D heat conduction problem in a rod of length l, whose ends are kept at zero temperature. Find the causal Green's function using:

   (a) Method of images;

   (b) Separation of variables and expansion in space eigenfunctions;

   (c) Laplace transform method.

   (There is no need to show equivalence of the three representations.)

5. The potential, Φ, due to a volume charge distribution, ρ, in three dimensional space, satifies the Poison equation $\nabla^2\Phi = -4\pi\rho$. Find the potential for the following two problems:

   (a) Volume charge distribution given by $\rho = \rho_0 \sin\alpha x \sin\beta y \sin\gamma z$, where $\rho_0,\ \alpha,\ \beta,\ \gamma$ are constants;

   (b) Surface charge distribution $\sigma = \sigma_0 \cos(\alpha x + \beta y)$ at $z = 0$. Note that at $z = 0$, the normal derivative of the solution has a jump equal to $4\pi\sigma$. Look for a solution that is finite as $|z| \to \infty$.

6. Consider the initial value problem

$$u_{tt} = \nabla^2 u + f(\mathbf{x})g(t), \quad |\mathbf{x}| < \infty, \quad t > 0$$

$$u = u_t = 0, \quad t = 0, \quad |\mathbf{x}| < \infty, \quad t > 0$$

$$u \quad \text{bounded as} \quad |\mathbf{x}| \to \infty.$$

   Here $\mathbf{x} = (x, y, z)$, $f(\mathbf{x})$ is non zero only in the compact region Γ, and $g(t) = \sin\omega t H(t)$, where $H(t)$ is the Heaviside function. Let $\mathbf{x}$ be a fixed point outside of Γ.

   (a) Show that $u(\mathbf{x}, t) \equiv 0$ for $t < T_1 < \infty$, where T_1 is the arrival time.

   (b) Show that $u(\mathbf{x}, t) = v(\mathbf{x}) \sin\omega t$ for $t > T_2$. That is u is a periodic after a finite amount of time.

   (c) Give a geometrical interpretation of T_1 and T_2.