Ph.D Qualifying Exam in Applied Mathematics

January 15, 2003. 10:00am-1:00pm.

Problem 1. After a drug is delivered to the bloodstream, its concentration C decreases with a rate which is proportional to the drug's concentration; the constant of proportionality k is called the assimilation rate. Assume that the drug is administered at regular time intervals, every T hours say, and that the concentration of each dosage is C_0. Assume also that delivery is instantaneous (i.e., the concentration increases by C_0 immediately) as soon as the drug is administered.

(a) What is the concentration of the drug, R_n, present in the blood after n dosages have been delivered? What is the level, R, in the limit $n \to \infty$?

(b) Sketch the concentration, C, of the drug versus time for two cases (i) $kT >> 1$, and (ii) kT of $O(1)$. Show that in the latter case the drug concentration for long times (i.e. as $n \to \infty$) oscillates between R and $C_0 + R$.

(c) Very often the goal is to keep $C(t)$ approximately between some lowest effective concentration, L, and some highest harmless concentration, H. Assume that $R = L$, $C_0 = H - L$, and find the time interval T so that this goal is achieved.

Problem 2. For the boundary value problem

$$Lu \equiv \left(-\frac{d^2}{dx^2} - \mu\right) u = f \quad x \in (0,1) \quad u(0) = c_0, \quad u(1) = c_1 \tag{1}$$

where μ is a parameter, write down the associated problem for the eigensystem (i.e., the eigenvalues and eigenfunctions) and construct the eigensystem. Use this to find the eigenfunction expansion of the solution to (1).

What happens, in general, to the solution as $\mu \to (n\pi)^2$, $(n = 1, 2, ...)$ and under what circumstances can this be avoided. Explain.

Problem 3. Construct the Green's function associated with the boundary value problem

$$Lu \equiv u'' - 2u' + u = f \quad x \in (0,1) \quad B_1 u \equiv u'(0) - 2u(0) = 0, \quad B_2 u \equiv u'(1) - (\frac{3}{2} + \epsilon)u(1) = 0 \tag{2}$$

for $\epsilon > 0$. What happens to the Green's function as $\epsilon \to 0$ and what happens to the solution u of (2) for general f? Explain, with no further calculation needed.

Problem 4. Use the method of images to solve for the Green's function $g(x,t|\xi,\tau)$ that satisfies

$$\begin{aligned}
&g_t - g_{xx} = \delta(x-\xi)\delta(t-\tau) \; ; \; x, \xi \in (0,L), \; -\infty < t, \tau < \infty \\
&g(x,t|\xi,\tau) = 0 \; ; \; x, \xi \in (0,L), \; t < \tau^-, \\
&g(0,t|\xi,\tau) = g(L,t|\xi,\tau) = 0 \; ; \; \xi \in (0,L), \; -\infty < t, \tau < \infty.
\end{aligned} \tag{3}$$

Problem 5. An *interior* two-dimensional problem satisfying Laplace's equation is defined in polar coordinates by the curves $\phi = 0$, $\phi = \beta$, and $\rho = \alpha$. A point source of strength -1 is placed in the interior region at (ρ', ϕ').

(a) Formulate the PDE problem satisfied by the Green's function for solving interior Dirichlet boundary value problems.
(*NOTE*: $\nabla^2 = \partial_{\rho\rho} + \frac{1}{\rho}\partial_\rho + \frac{1}{\rho^2}\partial_{\phi\phi} = \frac{1}{\rho}\partial_\rho(\rho\partial_\rho\ ) + \frac{1}{\rho^2}\partial_{\phi\phi}$.)

(b) Use separation of variables in polar coordinates to show that the solution of the problem in *a)* is

$$G(\rho,\phi|\rho',\phi') = \sum_{m=1}^{\infty} \frac{4}{m}\rho_<^{m\pi/\beta}\left(\frac{1}{\rho_>^{m\pi/\beta}} - \frac{\rho_>^{m\pi/\beta}}{\alpha^{2m\pi/\beta}}\right)\sin\left(\frac{m\pi\phi}{\beta}\right)\sin\left(\frac{m\pi\phi'}{\beta}\right), \tag{4}$$

where $\rho_< = min\{\rho,\rho'\}$ and $\rho_> = max\{\rho,\rho'\}$.

(c) Use Green's theorem and the above Green's function to show that the solution which reduces to zero on the curves $\phi = 0,\ \phi = \beta$, and to V_0 on $\rho = \alpha$, is given by

$$u(\rho,\phi) = 16V_0\sum_{m=0}^{\infty}\left(\frac{\rho}{\alpha}\right)^{(2m+1)\pi/\beta}\frac{\sin\frac{(2m+1)\pi\phi}{\beta}}{2m+1}. \tag{5}$$

Problem 6. Consider the following problem in a bounded domain Ω contained in $\mathbf{R}^3$:

$$\nabla^2 u = p(x,y,z)u, \quad (x,y,z)\in\Omega, \tag{6}$$

where $p(x,y,z)$ is suitably differentiable and satisfies $p(x,y,z) \geq 0, \quad (x,y,z)\in\Omega$.

We need to solve (6) subject to each of the following boundary conditions:

(a) $u = f(x,y,z)$ on $\partial\Omega$.

(b) $\frac{\partial u}{\partial n} = f(x,y,z)$ on $\partial\Omega$, where $\mathbf{n}$ denotes the outward normal to the domain.

(c) $\frac{\partial u}{\partial n} + \alpha u = f(x,y,z)$ on $\partial\Omega$, where $\alpha > 0$.

1. Given two suitably differentiable functions Φ and Ψ defined in Ω, prove the Green's identity

$$\int_\Omega \left[\nabla\Phi\cdot\nabla\Psi + \Phi\nabla^2\Psi\right] dV - \int_{\partial\Omega}\Phi\frac{\partial\Psi}{\partial n}dS = 0. \tag{7}$$

2. Using the result (7) and appropriately chosen Φ and Ψ, prove that the solution of (6) for each of the boundary conditions (a)-(c), is unique.

3. Taking $p = 0$, find the solution of (6) subject to the boundary conditions (i) $u = 1$ on $\partial\Omega$, and (ii) $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$? Are the solutions unique?