Doctoral Qualifying Exam, Analysis.

August 30, 1999.

1. Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2 x}.$$

For what values of x does the series converge absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous wherever the series converges? Is f bounded? (Justify your answers.)

2. Define the Gamma function by

$$\Gamma(y) = \int_0^{\infty} e^{-x} x^{y-1}\, dx.$$

(a) Prove that the integral exists as a Lebesgue integral for each real $y > 0$.

(b) Prove that $\Gamma(n+1) = n!$.

3. Consider the linear system

$$x = Ax + b$$

where x and b are n-vectors and $A = (a_{ij})$ is a $n \times n$ matrix which satisfies

$$\sum_{k=1}^{n} |a_{jk}| < 1$$

for $j = 1, \ldots, n$.

(a) Prove that the unique solution can be obtained as the limit of the iterative sequence

$$x^{m+1} = Ax^m + b.$$

(Hint: use a fixed point theorem for contractions, with metric $d(u,v) = \max_i |u_i - v_i|$ for n-vectors u and v.)

(b) Derive the error bound

$$d(x^m, x) \le \frac{\alpha^m}{1-\alpha} d(x^0, x^1)$$

where $0 \le \alpha < 1$.

4. Evaluate the following quantities using contour integration in the complex plane:

$$\text{(a)} \quad \int_0^{\infty} \frac{dx}{x^2+5x+6}, \qquad \text{(b)} \quad \int_{|z|=2} \frac{z \sin z}{\cos^3 z} dz, \qquad \text{(c)} \quad \sum_{n=0}^{\infty} \frac{1}{n^2+4}.$$

5. Given $f(z) = 1/[z(z-1)(z-3)]$, find the Laurent expansion about $z = 0$ in the regions: (a) $|z| < 1$, (b) $1 < |z| < 3$, (c) $|z| > 3$.

6. Classify all singular points, including branch points and the point at infinity for the following functions

$$\text{(a)} \quad w_a = \frac{1 - \cosh\sqrt{z}}{z}, \qquad \text{(b)} \quad w_b = iz + \sqrt{1 - z^2}.$$