Doctoral Qualifying Exam:
Real and Complex Analysis.

Thursday, September 3, 1998.

You have three hours for this exam. Show all working in the books provided.

1. Let

$$f_n(x) = \begin{cases} 0 & \left(x < \frac{1}{n+1}\right), \\ \sin^2 \frac{\pi}{x} & \left(\frac{1}{n+1} \leq x \leq \frac{1}{n}\right), \\ 0 & \left(\frac{1}{n} < x\right). \end{cases}$$

Sketch the function.
(a) Show that f_n converges pointwise to zero as $n \to \infty$.
(b) Show that the convergence is not uniform. Justify your answer.
(c) For what values of x does $\sum f_n$ converge? Justify your answer.

2. Let $\mathbf{I}$ be the set of irrational numbers. Define $f_n(x)$ by

$$f_n(x) = \begin{cases} n^2 & \text{for } x \in \left[\frac{1}{n+1}, \frac{1}{n}\right] \cap \mathbf{I}, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Is f_n Riemann integrable on $[0, 1]$ for each n? Explain.
(b) Does the set of points where the function $f_n(x)$ is discontinuous have zero measure? Explain.
(c) Evaluate the Lebesgue integral

$$\int_0^1 f_n(x)\, dx$$

(d) Show that $\lim_{n\to\infty} \int_0^1 f_n\, dx \neq \int_0^1 \lim_{n\to\infty} f_n\, dx$. Why doesn't the Lebesgue dominated convergence theorem apply in this case?

3. Suppose f is a continuously differentiable function with period 2π. Does the Fourier series of f converge uniformly for $x \in (0, 2\pi)$? If α/π is irrational, prove that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x + n\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, dt$$

for every x. *Hint:* Do it first for $f(x) = e^{ikx}$.

4. Evaluate the integral

$$\int_{Br} \frac{e^{st}}{(s-a)\sqrt{s}}\, ds$$

where $a > 0$ and the contour Br is any vertical line in the complex plane which lies to the right of $s = a$. Investigate the behavior of this integral as $a \to 0$.

5. Either By considering the function

$$g(z, \zeta) = \frac{\pi}{\sin(\pi z)(z^2 + \zeta^2)}$$

and a suitable contour, which you must define, find an explicit formula for the function $f(\zeta)$ defined by the infinite series

$$f(\zeta) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2 + \zeta^2}.$$

Deduce from this formula that

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}.$$

Or Evaluate the integral

$$\int_0^{\infty} \frac{x}{x^3 + 1}\, dx$$

6. Solve the boundary value problem

$$\begin{aligned}
\nabla^2 \phi &= 0, \qquad 0 < y < \infty, \quad |x| < \infty, \\
\frac{\partial \phi}{\partial y} &= 0, \qquad y = 0, \quad |x| > 1, \\
\phi &= x, \qquad y = 0, \quad |x| < 1
\end{aligned}$$

by using the appropriate conformal map. Leave your answer in terms of the transformed variable. What is the asymptotic behavior of ϕ as $r = \sqrt{x^2 + y^2} \to \infty$?