Doctoral qualifying exam:
Real and complex analysis.

September 2, 1997.

You have three hours for this exam. Show all working in the books provided.

1. Define f by

$$f(z) \;=\; \sum_{n=1}^{\infty} \frac{z^n}{n}$$

for all values of $z \in \mathbf{C}$ for which the series converges.
(a) Prove that f is defined if $|z| < 1$.
(b) Prove that for $|z| < 1$

$$f'(z) \;=\; \frac{1}{1-z}.$$

(c) Prove that for $|z| < 1$

$$f(z) \;=\; -\log(1-z).$$

(d) Is f defined for any z with modulus greater than or equal to one? Explain either why or why not, and if so, what is the value of f at those points?

2. Consider $F : \mathbf{R} \mapsto \mathbf{R}$ defined by $F(x) = 2 + x - \arctan x$.
(a) Show that $|F(x) - F(y)| < |x - y|$ for all $x, y \in \mathbf{R}$.
(b) Let $x_0 \in \mathbf{R}$. Discuss the behavior of the sequence defined by $x_{n+1} = F(x_n)$ for $n = 0, 1, 2, \ldots$.

3. Consider

$$F_n(x) = \log\left(x^2 + \frac{1}{n}\right)$$

which is defined for $x \in [-1, 1]$ and all $n \in \mathbf{N}$.
(a) Prove that F_n is Lebesgue integrable on $[-1, 1]$ for all $n \in \mathbf{N}$.
(b) Prove that the sequence of functions $F_1, F_2, F_3, \ldots$ does not converge uniformly on $[-1, 1]$.
(c) Prove that

$$\lim_{n\to\infty} \left(\int_{-1}^{1} F_n(x)\, dx \right) \;=\; \int_{-1}^{1} \left(\lim_{n\to\infty} F_n(x) \right)\, dx.$$

4. (a) Find all the Laurent series of the function $f(z) = \frac{1}{z+1}$ about the point $z = 1$. Where are these series valid?

(b) Use the principal branch of log, i.e., $\log w = \log|w| + i\arg(w)$ with $\arg(w) \in (-\pi, \pi)$, to show that

$$f(z) = \log\left(\frac{z}{z-1}\right)$$

is analytic for $|z| > 1$. Find the Laurent series of $f(z)$ about $z = 0$ in this region.

5. Find the singular points of the following functions. Classify them as removable, poles, or essential singularities, and give the residues there.

$$\frac{1}{z(z-1)^2} \qquad \frac{\sin z}{e^z - 1} \qquad z^2 \cos\left(\frac{1}{z}\right)$$

6. (a) Show that

$$\int_0^\infty \frac{\sin x}{x(x^2+1)}\,dx = \frac{\pi}{2}(1 - e^{-1}).$$

(Hint: integrate $f(z) = \frac{e^{iz}}{z(z^2+1)}$ around the contour consisting of: a large semi-circle $|z| = R$ in the upper half-plane, a small semi-circle $|z| = \epsilon$ in the upper half-plane, joined by straight line segments with $Im(z) = 0$ from $z = -R$ to $z = -\epsilon$ and from $z = \epsilon$ to $z = R$.

(b) Show that for real $a \in (-1, 1)$

$$\int_0^\infty \frac{x^a}{1+x^2}\,dx = \frac{\pi}{2\cos\frac{a\pi}{2}}.$$

(Hint: integrate $f(z) = \frac{z^a}{1+z^2}$ around the 'keyhole' contour with $\arg(z) \in (0, 2\pi)$ consisting of: straight line segments just above and just below the positive real axis from $z = \epsilon$ to $z = R$, joined by circular arcs $|z| = \epsilon$ and $|z| = R$.