Doctoral Qualifing Examination
in
Applied Mathematics

Part A: Analysis

September 16, 1996

Problem 1. Suppose $I = [0, \infty)$, $A = \bigcup_{n=0}^{\infty} (\mathbf{Q}^+ + \{\pi^n\})$, and

$$g(x) = \begin{cases} 1, & x \in A \\ e^{-[x]}, & x \in I - A, \end{cases}$$

where $[x]$ is the greatest integer $\leq x$.

1. Evaluate the Lebesgue integral $\int_I g(x)\, dx$.
2. Let $f_n \in L(I)$ for each $n = 1, 2, \dots$ and $f_n(x) \to f(x)$ a.e. on I.
   (a) Show that $\frac{g(x) \sin(1+f_n(x))}{1+x^2} \in L(I)$.
   (b) Find $\lim_{n \to \infty} \int_I \frac{g(x) \sin(1+f_n(x))}{1+x^2}\, dx$.

Problem 2. Find the sum of the Fourier series (do not compute the coefficients of the series) of a 2π-periodic function $f(x)$ if

$$(a) \quad f(x) = \begin{cases} 0, & -\pi \leq x < 0 \\ x, & 0 < x < \pi, \end{cases} \qquad\qquad (b) \quad f(x) = x^{2/3} \quad \text{for} \quad x \in [-\pi, \pi).$$

Problem 3.

1. Show that the Lebesgue integral

$$F(p) = \int_1^{\infty} \frac{\log x}{(x-1)x^p}\, dx = \sum_{n=0}^{\infty} \frac{1}{(n+p)^2}, \qquad p > 0.$$

2. Find $F'(p)$ in two ways: in the integral form and in the series form.

Problem 4.

1. Find all possible values of i^{-i}.

2. If z_1, z_2 and z_3 all lie on the unit circle show that

$$\arg\left(\frac{z_3 - z_2}{z_3 - z_1}\right) = \frac{1}{2}\arg\frac{z_2}{z_1}.$$

Problem 5.

1. Describe all branches (Riemann Sheets) of $f(z) = \sqrt{i + \sqrt{z}}$. Can any of the branches be expanded in a Taylor Series about the point $z = -1$? If so, what is the radius of convergence of the series?

2. How many roots of the equation $z^4 - 5z + 1 = 0$ lie inside the unit circle $|z| < 1$? How many roots of this equation lie in the right half plane?

Problem 6. Evaluate each of the following integrals:

$$(a) \quad \int_0^\infty \frac{\sin ax}{x(1+x^2)}\, dx, \qquad a \text{ real}, \qquad\qquad (b) \quad \int_0^1 \frac{\sqrt{x}\sqrt{1-x}}{(1+x)^2}\, dx.$$