Doctoral Qualifying Examination

in

Applied Mathematics

Part A: Analysis

September 18, 1995

Problem 1. Suppose that $f : \mathbf{R} \to \mathbf{R}$ is Lebesgue integrable with

$$\int f(x)\,dx \;=\; 1.$$

Show that if $\phi : \mathbf{R} \to \mathbf{R}$ is bounded and continuous then

$$\lim_{\alpha \to \infty} \alpha \int f(\alpha x)\phi(x)\,dx \;=\; \phi(0).$$

Problem 2. Show that each of the following series converges uniformly on every compact subset of $(0, 2\pi)$ to a function of x that is differentiable on $(0, 2\pi)$. In which cases can the derivative be determined using term-by-term differentiation?

$$\text{(A)}\ \sum_{n=0}^{\infty} \left(\frac{x}{\pi} - 1\right)^{2n} \qquad \text{(B)}\ \sum_{n=0}^{\infty} \frac{1}{n} \sin(nx) \qquad \text{(C)}\ \sum_{n=0}^{\infty} \frac{1}{n} \sin\left[\frac{(n^3+1)x}{n^2}\right]$$

Problem 3. Let f_1 and g_1 be continuous real-valued functions defined on $[-1, 1]$. Consider the sequences $\{f_n\}$ and $\{g_n\}$ given by

$$\begin{aligned} f_{n+1}(x) &= \sin x \;+\; \frac{g_n(x) + g_n(-x)}{2} \;+\; \frac{1}{5}\int_{-1}^{1} e^{-|x-y|} g_n(y)\,dy \\ g_{n+1}(x) &= \cos x \;+\; \int_{-x}^{x} y f_n(y)\,dy. \end{aligned}$$

Show that there exist continuous functions f and g defined on $[-1, 1]$ such that $f_n \to f$ and $g_n \to g$ pointwise.

Problem 4. The expressions below define functions on the interval $[0, 2\pi]$. For each of these functions determine the points $x \in [0, 2\pi]$ where the fourier series of the function converges to the value of the function at x.

$$\text{(A) } e^{-x} \qquad \text{(B) } \sin(\frac{1}{2}x) \qquad \text{(C) } \begin{cases} x^2 - 1 & x \in (1,2) \cap \mathbf{Q} \\ x & x \notin \mathbf{Q} \\ \pi & \text{otherwise} \end{cases}$$

Problem 5. Define the branch of $w = \sqrt{z+1}$ such that $\operatorname{Re} w > 0$ on one of the sheets.

(A) How many Riemann sheets does w have?
(B) With the branch cut defined such that $\operatorname{Re} w > 0$ on one of its sheets, what can be said about the $\operatorname{Re} w$ on the remaining sheets?

Problem 6. Find all the branches of $w = \sqrt{1 + \sqrt{z}}$.

Problem 7. Is $u = (x^2 + y^2)^{1/4} \cos\left(\frac{1}{2} \tan^{-1}(y/x)\right)$ harmonic? On what domain?

Problem 8. Evaluate for $a > 0$, $a = 0$, and $a < 0$:

$$I(a) = \int_{-\infty}^{+\infty} \frac{e^{iax}}{1 + x^2}\, dx.$$

On what domain is $I(a)$ a complex analytic function of a?

Problem 9. State the maximum modulus theorem. Find the point(s) where the modulus of $f(z) = z^3 - 1$ attains is maximum on $|z| \leq 1$.