Doctoral Qualifying Examination

in

Applied Mathematics

Part A: Analysis

September 12, 1994

Problem 1.

(a) Suppose that $u_n : \mathbf{R} \to \mathbf{R}$ and that u_n is continuous for $n = 1, 2, \ldots$. Prove that if $u_n \to u$ uniformly then u is continuous.

(b) Let

$$f(x) \;=\; \sum_{n=1}^{\infty} \frac{(-1)^n \sin(nx)}{n}.$$

Show that $f(x)$ exists and is finite for all $x \in (-\pi, \pi)$ and that f is continuous on $(-\pi, \pi)$.

Problem 2. This problem concerns the question of under what conditions does the Fourier series of a function converge to the function. For a Lebesgue integrable function $f : [0, 2\pi] \to \mathbf{R}$, the Fourier coefficients of f are defined by

$$\hat{f}_n \;=\; \int_0^{2\pi} f(x) e^{-inx}\, dx.$$

For each function given below, evaluate for all $x \in [0, 2\pi]$ the Fourier series of f given by the sum

$$\frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \hat{f}_n e^{inx}.$$

(a) $f(x) \;=\; x$

(b)

$$f(x) \;=\; \begin{cases} 1/x & \text{if } x \text{ is rational} \\ 1/\sqrt{x} & \text{if } x \text{ is irrational and } x < 1/2 \\ 1 & \text{otherwise} \end{cases}$$

(c) $f(x) = \lim_{n\to\infty} f_n(x)$ where

$$f_n(x) \;=\; \begin{cases} n & x \in (0, 1/n) \\ 0 & x \notin (0, 1/n) \end{cases}$$

(d) $f(x) \;=\; \exp(\cos x - \sin^4 x)$

Problem 3. Suppose that $f : \mathbf{R} \to \mathbf{R}$ and $K : \mathbf{R}^2 \to \mathbf{R}$ are continuous. Show that there exists a unique function $\varphi : \mathbf{R} \to \mathbf{R}$

$$\varphi(x) \;=\; f(x) \;+\; \int_0^x K(x,y)\varphi(y)\,dy$$

for all $x \in \mathbf{R}$.

Problem 4. Find all possible values of

$$(1+i)^{1-i}.$$

Problem 5. For each function given below, (a) determine the largest region where it is analytic, (b) identify all branch cuts, (c) classify all singularities, (d) give the lowest order term in the Laurent series about each pole, (e) determine the radius of convergence of the Taylor series about the point $z = 2$. Be sure to consider all sheets of multivalued functions.

$$\begin{aligned}
(a)\quad f(z) &= \frac{1}{1-z} \\
(b)\quad f(z) &= \sqrt{z} \\
(c)\quad f(z) &= \frac{1}{1-\sqrt{z}} \\
(d)\quad f(z) &= \cos\sqrt{z} \\
(e)\quad f(z) &= \left(x + \frac{x}{x^2+y^2}\right) \;+\; i\left(y - \frac{y}{x^2+y^2}\right) \qquad (x = \Re z \text{ and } y = \Im z)
\end{aligned}$$

Problem 6. Compute each of the following definite integrals:

$$\text{(a)}\ \int_0^\infty \frac{1}{1+x^4}\,dx \qquad \text{(b)}\ \int_0^\infty \frac{\sqrt{x}}{1+x^4}\,dx \qquad \text{(c)}\ \int_0^\infty \frac{\cos(tx)}{1+x^2}\,dx \quad (t \in \mathbf{R})$$