Ph.D Qualifying Exam in Analysis

August 29, 2001

Problem 1. Let f be a non-negative measurable function on a measure space (X, M, μ), and define $\lambda(E) = \int_E f d\mu$ for $E \in M$.

(a) Prove λ is a measure on M.

(b) Prove that if g is a non-negative measurable function on (X, M, μ) that $\int g d\lambda = \int f g d\mu$. (Hint: Prove the result first for g a simple function, then use the monotone convergence theorem.)

Problem 2. Let $f_n(x) = x/(1 + nx^2)$ if $x \in \mathbb{R}$, $n = 1, 2, ...$ Find the limit function f of the sequence $\{f_n\}$ and the limit function g of the sequence $\{f_n'\}$.

(a) Prove that $f'(x)$ exists for every x, but that $f'(0) \neq g(0)$. For what values of x does $f'(x) = g(x)$?

(b) On what subintervals of $\mathbb{R}$ does $f_n \to f$ uniformly?

(c) On what subintervals of $\mathbb{R}$ does $f_n' \to g$ uniformly?

Problem 3. Establish the validity of the following formulas:

$$\text{(a) } x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sin nx}{n} \quad \text{if } -\pi < x < \pi.$$

$$\text{(b) } x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos nx}{n^2} \quad \text{if } -\pi \le x \le \pi.$$

Problem 4. (a) Use Liouville's theorem to show that any nth degree polynomial $P_n(z)$, $n \geq 1$, has a zero.
(b) Prove the result in (a), but this time use the maximum modulus theorem.

Problem 5. Evaluate the following integrals:

$$\text{(a) } \int_{|z|=2} z e^{3/z} dz, \quad \text{(b) } \int_{|z|=1} \frac{dz}{(1-e^{-z})^n}, \text{ where } n \text{ is a positive integer,}$$

$$\text{(c) } \int_{-\infty}^{\infty} \frac{\cos x - \cos a}{x^2 - a^2} dx, \ a \in \mathbb{R}.$$

Problem 6. (a) Find the number of zeros of $f(z) = z^4 - 5z + 1$ in $1 \le |z| \le 2$.
(b) Suppose that f is entire and $f(z)$ is real if and only if z is real. Use the argument principle to show that f can have at most one zero.