Ph.D Qualifying Exam in Analysis

August 26, 2002

Problem 1. Let $\{r_n : n \geq 1\}$ be the set of all rational numbers in the interval $[0,\infty) \subset \mathbb{R}$ and define the sequence of functions $\{f_n : n \geq 1\}$ on $[0,\infty)$ as follows:

$$f_n(x) := \begin{cases} 0, & x \in [0,1] \cup \{r_1, ..., r_n\} \\ 1/x^2, & x \notin [0,1] \cup \{r_1, ..., r_n\} \end{cases}.$$

(a) Compute $f := \lim_{n\to\infty} f_n$.

(b) Is the convergence in (a) uniform anywhere? Explain.

(c) Is f (Cauchy) Riemann integrable on $[0,\infty)$? Explain.

(d) Prove that $f \in L^1([0,\infty))$ and show how one can use Lebesgue's dominated convergence theorem to compute $\int_0^\infty f dx$.

Problem 2. Use Fourier analysis to prove that

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n} > 0, \quad \forall\, 0 < x < \pi,$$

and cite all the necessary convergence theorems. (Hint: find a suitable function on $[0,2\pi]$ whose Fourier series is the given series)

Problem 3. By using the counting measure and the appropriate fundamental theorems, prove that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} < \infty \Longrightarrow \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{mn} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn},$$

when $\forall\, a_{mn} \geq 0$.

Problem 4. Derive a formula for $w = \arccos(z)$ and discuss the branch point/branch cut structure of this function. Be certain to include the point at infinity in your discussion.

Problem 5. Find the maximum of $|f(z)|$ in $|z-2| \leq 1$ for $f(z) = z^2 - 7z + 13$.

Problem 6. Evaluate the following using complex analysis techniques:

(a) $\displaystyle\int_{|z|=2} \frac{1}{z(e^{6z}-1)}\, dz$

(b) $\displaystyle\int_0^\infty \frac{\ln^2(x)}{x^2+1}\, dx$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$