Ph.D Qualifying Exam in Analysis

May 11, 2000

Problem 1. Let $X = N$, $\mathbf{X}$ = power set of N, and μ be the counting measure on $\mathbf{X}$. Suppose f is a nonnegative function on $X = N$.

(a) Prove $f \in M^+(X, \mathbf{X})$.

(b) Prove $\int f d\mu = \sum_{n=1}^{\infty} f(n)$.

(c) If $f \in L_p$ and $1 \leq p \leq s < \infty$, show that $f \in L_s$.

Problem 2. If $\{x_n\}$ is an increasing sequence of points in $[a, b]$ and if $\sum |c_n|$ converges, prove $f(x) = \sum_{n=1}^{\infty} c_n \chi_{[a,b]}(x)$ converges uniformly and that f is continuous for all $x \neq x_n$.

Problem 3. Let $f(x) = (\pi - x)/2$, $x \in (0, 2\pi)$ and f be $2\pi-$periodic.

(a) Compute the Fourier series generated by f to show that $\sum_{n=1}^{\infty} (\sin nx)/n$ converges on $(0, 2\pi)$. Be sure to justify the convergence.

(b) Use part (a) and Parseval's equality to prove $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$.

(c) Use parts (a) and (b) to show that on $(0, 2\pi)$,

$$\sum_{n=1}^{\infty} \frac{\cos nx}{n^2} = \frac{x^2}{4} - \frac{\pi x}{2} + \frac{\pi^2}{6}.$$

Does the result also hold at $x = 0, 2\pi$? Why?

Problem 4. Define the residue at infinity by

$$Res\,(f(z); \infty) = \frac{1}{2\pi i} \int_{C_\infty} f(z) dz,$$

where C_∞ denotes the limit $R \to \infty$ of a circle of radius R. Also, define f to be analytic at infinity if $f(1/t)$ is analytic at $t = 0$.

(a) Show that if f is analytic at ∞ with $f(\infty) = 0$, then f has the expansion

$$f(z) = \frac{a_{-1}}{z} + \frac{a_{-2}}{z^2} + \dots .$$

(b) Given the assumptions in (a), show that $Res(f(z); \infty) = \lim_{z \to \infty} z f(z)$.

(c) Show that for every rational function $f(z)$,

$$Res\,(f(z); \infty) = \sum_{j=1}^{N} Res(f(z); z_j),$$

where $z_1, \dots, z_N$ denote the singularities of f.

(d) Use the above ideas to compute

$$\int_C \frac{1+2z^2+3z^3+4z^4+5z^5}{1+z+27z^6}dz,$$

where C is a circle of radius 2.

Problem 5.

(a) A nonconstant function $F(z)$ is such that $F(z+a) = F(z)$, $F(z+bi) = F(z)$ for all z, where $a, b > 0$ are given constants. Prove that $F(z)$ cannot be analytic in the rectangle $0 \le x \le a, 0 \le y \le b$.

(b) Let f be analytic in a neighborhood of the closed unit disk. If $|f(z)| < 1$ for $|z| = 1$, show that there is a unique z with $|z| < 1$ and $f(z) = z$. If $|f(z)| \le 1$ for $|z| = 1$, what can you say? If $|f(z)| = 1$ for $|z| = 1$, find a formula for f.

Problem 6. Evaluate the following using contour itegration. (Justify your results.)

$$(a) \quad \int_0^\infty \frac{x^{\alpha-1}}{1+x}dx \quad (0<\alpha<1) \qquad (b) \quad \sum_{n=1}^{\infty} \frac{1}{1+n^2}.$$