Doctoral qualifying exam:
Real and complex analysis.

January 20, 1999.

You have three hours for this exam. Show all working in the books provided.

1. (a) Assume that $f_n \to f$ uniformly on a set S and that each f_n is bounded on S. Prove that $\{f_n\}$ is uniformly bounded on S. (Definition: $\{f_n\}$ is uniformly bounded if there exists an M such that $|f_n(x)| < M$ for all $x \in S$ and all n).
(b) Let

$$f_n(x) = \begin{cases} \frac{\sin\ x}{\sqrt{n}} \frac{1-\cos\ nx}{1-cos\ x} & \text{for } 1/n \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(i) What is $\lim_{n\to\infty} f_n$?
(ii) Show that f_n is not uniformly bounded on [0,1]. (Hint: look at $x = \pi/(2n)$).

2. Let $\{r_n\}$ be a listing of the rationals on $[0, 1]$. Define $f_n(x)$ by

$$f_n(x) = \begin{cases} 1 & \text{if } x = r_m \text{ with } m \leq n \\ 0 & \text{otherwise.} \end{cases}$$

(a) Is f_n Riemann integrable on [0,1] for each n? Explain. If so, evaluate the integral.
(b) Does the set $\{r_n\}$ have measure zero?
(c) What is $\lim_{n\to\infty} f_n$?
(d) Is $f(x) = \lim_{n\to\infty} f_n$ Riemann integrable on [0,1]? Is it Lebesgue integrable? Explain.

3. Let f be continuous on $[0, 2\pi]$ and periodic with period 2π. Consider the Fourier series generated by f

$$f(x) \sim \sum_{n=-\infty}^{\infty} a_n e^{inx}$$

with

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} f(t) e^{-int}\ dt.$$

Assume also that f' is Riemann integrable on $[0, 2\pi]$ and in $L^2[0, 2\pi]$ so that the Fourier series generated by f' is well defined (although it may not converge).
(a) Use Bessel's inequality to prove that $\sum_{n=-\infty}^{\infty} n^2 |a_n|^2$ converges.
(b) Use (a) and the Cauchy-Schwartz inequality to deduce that $\sum_{n=-\infty}^{\infty} |a_n|$ converges.
(c) From (b), deduce that the series $\sum_{n=-\infty}^{\infty} a_n e^{inx}$ converges uniformly to a continuous sum function g on $[0, 2\pi]$.

4. Show that the function $u(x,y) = x^3 - 3xy^2$ is harmonic and find $v(x,y)$, the harmonic conjugate of u. Show that $f = u + iv$ is an analytic function of the complex variable $z = x + iy$.

5. Expand the function

$$f = \frac{1}{z(z-1)(z-2)}$$

in a power series valid for
(i) $0 < |z| < 1$.
(ii) $1 < |z| < 2$.

6. Let $f(z)$ be a polynomial of degree n and $f(0) \neq 0$. Let

$$I = \frac{1}{2\pi i} \int_{C_R} z \frac{f'(z)}{f(z)} \, dz$$

where the prime denotes differentiation and C_R is the circle $|z| = R$. Show that

$$I = \sum_{k=1}^{n} z_k$$

where the z_k are the roots of f.

7. Find a conformal mapping which maps the first quadrant $0 < arg z < \pi/2$, $|z| > 0$ onto the interior of the unit circle $|w| < 1$ in such a way that $i, 0, 1$ map onto $-1, -i, 1$, respectively.