Doctoral Qualifying Exam:
Real and Complex Analysis.

Friday, January 23, 1998.

You have three hours for this exam. Show all working in the books provided.

1. Define f by

$$f(r,\theta) = \sum_{n=1}^{\infty} \frac{r^n e^{in\theta}}{n}$$

for all values of $(r,\theta) \in \mathbf{R}^2$ for which the series converges.
(a) Determine the values of r and θ for which the series converges and prove convergence for these values.
(b) Discuss the validity of the formula

$$\frac{\partial f}{\partial \theta} = i \sum_{n=1}^{\infty} r^n e^{in\theta}.$$

2. Consider $F : \mathbf{R} \mapsto \mathbf{R}$ defined by $F(x) = \alpha + x - \arctan x$ for $\alpha \in \mathbf{R}$. (a) Show that $|F(x) - F(y)| < |x - y|$ for $x \neq y$.
(b) Let $x_0 \in \mathbf{R}$. Discuss the behavior of the sequence defined by $x_{n+1} = F(x_n)$ for $n = 0, 1, 2, \ldots$.

3. Define

$$g(x) = \begin{cases} 0 & x \in \mathbf{Q} \\ x^2 & x \in \mathbf{R} - \mathbf{Q} \end{cases}$$

Prove that g is Lebesgue integrable on $[0,1]$ but is not Riemann integrable on this interval.

4. (a) Give the type of the singular points of the following functions in the complex plane. Include branch points and the point at infinity, and give the order of any poles

$$\frac{z^3}{z^2+z+1}, \qquad e^{\tan z}, \qquad \log(1+z^{\frac{1}{2}}).$$

(b) Show that the function $f(z) = |z|$ is not analytic at $z = 0$.

5. Let $f(t,z) = \exp(\frac{t}{2}(z - \frac{1}{z}))$ have the Laurent series

$$e^{\frac{t}{2}(z-1/z)} = \sum_{n=-\infty}^{\infty} J_n(t) z^n.$$

Show from the definition of a Laurent series that the Laurent coefficients $J_n(t)$ are given by

$$J_n(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(n\theta - t\sin\theta)}\,d\theta = \frac{1}{\pi}\int_0^{\pi}\cos(n\theta - t\sin\theta)\,d\theta.$$

Show that $f(t,z)$ satisfies

$$t^2\frac{\partial^2 f}{\partial t^2} + t\frac{\partial f}{\partial t} + t^2 f = z\frac{\partial}{\partial z}\left(z\frac{\partial f}{\partial z}\right)$$

and hence that $J_n(t)$ satisfies

$$t^2 J_n''(t) + tJ_n'(t) + (t^2 - n^2)J_n(t) = 0.$$

(The coefficients a_n in the Laurent series $f(z) = \sum_{-\infty}^{\infty} a_n(z-z_0)^n$ of $f(z)$ about $z = z_0$ are given by

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

where C is any closed contour around $z = z_0$ in the region of analyticity of $f(z)$.)

6. (a) Show that

$$\int_0^{\infty}\frac{\cos kx - \cos mx}{x^2}\,dx = -\frac{\pi}{2}(|k| - |m|).$$

(Hint: integrate $f(z) = (e^{ikz} - e^{imz})/z^2$ around a closed semi-circle in the upper half z-plane with a semi-circular indentation at $z = 0$.)

(b) Show that for $a > 0, t > 0$

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{zt}}{z^{1/2}}\,dz = \frac{1}{\sqrt{\pi t}},$$

where the contour is the line $Re(z) = a$ in the complex z-plane. First, deform the contour to a large semi-circle in the left half z-plane with a path around the branch cut of $z^{\frac{1}{2}}$. What happens if $a > 0$ but $t < 0$?